An Approximate Analysis of Laminar Free Convection on a Vertical Plate using von Kármán–Pohlhausen Method: Uniform Heat Flux

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Abstract

This study presents a robust analytical formulation of laminar free convection over a vertical flat plate with uniform heat flux, using the von Kármán–Pohlhausen integral method. By assuming the polynomial profiles for velocity and temperature and making use of the Boussinesq approximation, together with the integral momentum and energy equation, expressions were derived and simplified into first-order differential equations. Through a power-law solution approach, expressions for the boundary layer thickness, velocity scale, local heat transfer coefficient, and Nusselt number were obtained in terms of the Rayleigh and Prandtl numbers. The analytical results were later verified using a practical example of air flowing over a heated vertical plate, which strongly aligned with analytical results, thus confirming both the accuracy and computational efficiency of the method. MATLAB visualizations of Grashof, Rayleigh, and Nusselt numbers, heat flux, and boundary layer development were devised which enabled reproducibility . The document also focuses on drawing comparison between heat transfer through conduction and convection as well as spatially varying the heat flux

Keywords: Free convection; von Kármán–Pohlhausen method; vertical plate; boundary layer analysis; Nusselt number; MATLAB visualization

Consider the integral and momentum and energy equations by assuming velocity and temperature profiles. Consider the integral momentum equation as:

$$-\tau_w = \frac{1}{R}\frac{d}{dx}(R\int_0^Y \rho u^2 dy) - \frac{U_\infty}{R}\frac{d}{dx}(R\int_0^Y \rho u 2dy) + v_w\rho_w U_\infty + \int_0^Y (-\rho U_\infty \frac{dU_\infty}{dx})dy \tag{1}$$

The said procedure has been highlighted in [6]. By considering the body force term and adding it yo the momentum equation involves assuming $U_\infty = 0, v_w = 0$ and $R =$ constant, and along the added body force term, the integral term reduces to the following expression:

$$-\tau_w = \frac{d}{dx}(\int_0^Y \rho u^2 dy) + Y\frac{dp}{dx} + \int_0^Y \rho g dy \tag{2}$$

The pressure gradient term is given as:

$$\frac{dp}{dx} = -\rho_\infty g \tag{3}$$

By substitution of in results in:

$$\begin{aligned} -\tau_w &= \frac{d}{dx}\left(\int_0^Y \rho u^2\, dy\right) - Y\rho_\infty g + \int_0^Y \rho g\, dy \\ &= \frac{d}{dx}\left(\int_0^Y \rho u^2\, dy\right) - \int_0^Y \rho_\infty g\, dy + \int_0^Y \rho g\, dy \\ &= \frac{d}{dx}\left(\int_0^Y \rho u^2\, dy\right) - g\int_0^Y (\rho_\infty - \rho)\, dy. \end{aligned} \tag{4}$$

Using Boussinesq approximation, the density difference can be approximated as:

$$\rho_\infty - \rho \approx \beta\rho(T - T_\infty) \tag{5}$$

The substitution of in 4 results in:

$$-\tau_w = \frac{d}{dx}(\int_0^Y \rho u^2\, dy) - g\beta\rho \int_0^Y (T - T_\infty)\, dy \tag{6}$$

or

$$\rho\frac{d}{dx}(\int_0^\delta u^2\, dy) = -\mu(\frac{\partial u}{\partial y})_{y=0} + \rho g\beta \int_0^\delta (T - T_\infty)\, dy \tag{7}$$

The above equation validates both for laminar and turbulent flows. The velocity gradient about the edge of boundary layer tends to vanish, i.e $\partial u(x,\delta)/\partial y = 0$. This means that the shear force term vanishes. Considering the integral energy equation:

$$q_w^{"} = \frac{1}{R}\frac{d}{dx}[R\int_0^Y \rho u(h^* - h_\infty^*)\, dy] \tag{8}$$

The applicable energy equation can be written thus, while assuming that $h^* - h_\infty^* = c_p(T - T_\infty)$ and hence the Boussinseq approximation can be written as:

$$q_w^{"} = \rho c_p \frac{d}{dx}[\int_0^Y u(T - T_\infty)\, dy] \tag{9}$$

or

$$\rho c_p \frac{d}{dx}[\int_0^Y u(T - T_\infty)\, dy] = -k(\frac{\partial T}{\partial y})_{y=0} \tag{10}$$

It is generally assumed that the velocity and thermal boundary layers have same local thickness. The assumptions taken into account for deriving then integral formulation of energy equation are as follows:

- No changes in kinetic and potential energy
- There is negligible conduction in axial direction
- No viscous dissipation
- Constant fluidic properties

During the integral analysis, it is assumed that $\delta \approx \Delta$, which is justified through the results obtained with this assumption that agrees quite nicely with the experimental work of Touloukian et al. (1948). Sparrow and Gregg (1956) performed analysis for $\delta \neq \Delta$ and the results currently available are close to their work. The following approach requires less work and computational power and effort. For detailed analysis of integral analysis, can be referred to Kakac et al. (2014) and Ghiaasiaan (2018)

Uniform Heat Flux

Some technical applications consider the heat flux to be prescribed over the entire surface where it is deemed necessary to evaluate the temperature distribution about the entire surface, as discussed in Sparrow (1955). By assuming laminar boundary layer flow over a vertical plate and third-degree velocity polynomial as:

$$u(x,y) = a_o(x) + a_1(x)y + a_2(x)y^2 + a_3(x)y^3 \tag{11}$$

The corresponding boundary conditions are:

$$u(x,0) = 0 \tag{12}$$

$$u(x,\delta) = 0 \tag{13}$$

$$\frac{\partial u(x,\delta)}{\partial y} = 0 \tag{14}$$

$$\frac{\partial^2 u(x,\delta)}{\partial y^2} = \frac{-\beta g}{\nu}(T_w - T_\infty) \tag{15}$$

The is obtained from the x-momentum equation by setting y=0. Using the very boundary conditions, the following coefficients are obtained:

$$a_o = 0$$
$$a_1 = \frac{\beta g(T_w - T_\infty)}{4\nu}\delta$$
$$a_2 = \frac{-\beta g(T_w - T_\infty)}{2\nu}$$
$$a_3 = \frac{-\beta g(T_w - T_\infty)}{4\nu}\frac{1}{\delta}$$

The substitution of a_o, a_1, a_2 and a_3 result in the following velocity profile:

$$u = \frac{\beta g(T_w - T_\infty)}{4\nu}\delta y[1 - 2(\frac{y}{\delta}) + (\frac{y}{\delta})^2] \tag{16}$$

It can be written as:

$$u = u_c(\frac{y}{\delta})[1 - \frac{y}{\delta}]^2 \tag{17}$$

$\delta(x)$ ad $u_c(x)$ are unknown and u_c is evaluated as:

$$u_c = [\frac{\beta g(T_w - T_\infty)}{4\nu}\delta^2] \tag{18}$$

The next step involves determining the temperature profile, for said purpose, a second-degree polynomial shall be used:

$$T(x, y) = b_o(x) + b_1(x)y + b_2(x)y^2 \tag{19}$$

The corresponding boundary conditions are:

$$T(x, \Delta) = T_\infty \tag{20}$$

$$\frac{\partial T(x, \Delta)}{\partial y} = 0 \tag{21}$$

$$\frac{\partial T(x, 0)}{\partial y} = -\frac{q_w^{"}}{k} \tag{22}$$

The unknown coefficients b_o, b_1 and b_2 are evaluated as:

$$b_o = T_\infty + \Delta(\frac{q_w^{"}}{2k})$$

$$b_1 = (-\frac{q_w^{"}}{k})$$

$$a_2 = (\frac{q_w^{"}}{2k}\frac{1}{\Delta})$$

The substitution of b_0, b_1 and b_2 in results in the following expression:

$$T - T_\infty = (\frac{q_w^{"}}{2k})\delta(1 - \frac{y}{\Delta})^2 \tag{23}$$

Since $\delta \approx \Delta$ thus assuming uniform heat flux, 17 and 23 are introduced into integral momentum and energy equation which upon evaluation result in following expressions :

Evaluating the integrals and rearranging them yield

$$\frac{1}{105}\frac{\mathrm{d}}{\mathrm{d}X}\left[\Omega^2\Delta\right] = \frac{\Delta^2}{6} - \frac{\Omega}{\Delta}. \tag{24}$$

$$\frac{1}{30}\frac{\mathrm{d}}{\mathrm{d}X}\left[\Omega\Delta^2\right] = \frac{2}{Pr} \tag{25}$$

Where

$$X = x(\frac{g\beta q_w^{"}}{k\nu^2})^{\frac{1}{4}}$$

$$\Delta = \delta(\frac{g\beta q_w^{"}}{k\nu^2})^{\frac{1}{4}}$$

$$\Omega = u_c(\frac{g\beta q_w^{"}\nu^2}{k})^{\frac{-1}{4}}$$

The solutions of 24 and 25 are as follows:

$$\Omega = (6000)^{\frac{1}{5}} Pr^{\frac{-1}{5}} (0.8 + Pr)^{\frac{-2}{5}} X^{\frac{3}{5}} \tag{26}$$

$$\Delta = (360)^{\frac{1}{5}} Pr^{\frac{-2}{5}} (0.8 + Pr)^{\frac{1}{5}} X^{\frac{1}{5}} \tag{27}$$

Practical Example

The surface temperature of a square plate is maintained at a uniform surface heat flux $q^{"}$ instead of fixed surface temperature. The one side of the plate is 0.5m, and the plate is in vertical position in quiescent air at 1 atm and 300 K. Determine:
(a) Determine the velocity boundary layer thickness at the trailing edge.
(b) Determine the total heat loss from the plate. Onur (2023)
Solution:
Solution:

A vertical square plate ($0.5\,\mathrm{m} \times 0.5\,\mathrm{m}$) is maintained at a uniform surface heat flux q''. The plate is exposed to quiescent air at atmospheric pressure and $T_\infty = 300\,\mathrm{K}$. The dimensions are:

$$L = W = 0.5\,\text{m}$$

We aim to determine:

(a) Velocity Boundary Layer Thickness at the Trailing Edge:

To proceed, we must estimate the average surface temperature T_s, which is not specified directly. For an initial approximation, assume:

$$T_s \approx 400\,\text{K} \Rightarrow T_f = \frac{T_s + T_\infty}{2} = \frac{400 + 300}{2} = 350\,\text{K}$$

From air property tables at $T_f = 350\,\text{K}$, we obtain:

$$\nu = 20.92 \times 10^{-6}\,\text{m}^2/\text{s}, \quad k = 30 \times 10^{-3}\,\text{W/mK}, \quad Pr = 0.700$$

The coefficient of thermal expansion is:

$$\beta = \frac{1}{T_f} = \frac{1}{350} = 2.857 \times 10^{-3}\,\text{K}^{-1}$$

Grashof number at the trailing edge ($x = L$) is estimated as:

$$\text{Gr}_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2} = \frac{9.81 \cdot 2.857 \times 10^{-3} \cdot (400 - 300) \cdot (0.5)^3}{(20.92 \times 10^{-6})^2} = 8.0 \times 10^8$$

Using empirical correlations for natural convection, the velocity boundary layer thickness is:

$$\delta = 3.93 \left[\frac{Pr + 0.952}{Pr^2 \cdot Gr_L}\right]^{1/4} L = 3.93 \left[\frac{0.7 + 0.952}{(0.7)^2 \cdot 8.0 \times 10^8}\right]^{1/4} (0.5) = 0.01582\,\text{m}$$

(b) Total Heat Loss from the Plate:

The Rayleigh number is:

$$Ra_L = Gr_L \cdot Pr = 8.0 \times 10^8 \cdot 0.7 = 5.6 \times 10^8$$

For a vertical plate with constant heat flux and laminar flow, the average Nusselt number can be expressed as (for $10^4 < Ra_L < 10^9$):

$$\overline{Nu}_L = 0.59 \cdot Ra_L^{1/4} = 0.59 \cdot (5.6 \times 10^8)^{1/4} = 74.84$$

The average heat transfer coefficient is:

$$\overline{h} = \frac{k}{L} \cdot \overline{Nu}_L = \frac{30 \times 10^{-3}}{0.5} \cdot 74.84 = 4.49\,\text{W/m}^2\text{K}$$

Assuming the estimated temperature difference $(T_s - T_\infty) = 100\,\text{K}$, the total heat loss from both sides of the plate is:

$$q = 2 \cdot \overline{h} \cdot A \cdot (T_s - T_\infty) = 2 \cdot 4.49 \cdot (0.5 \cdot 0.5) \cdot (400 - 300) = 224.5\,\text{W}$$

If the surface heat flux q'' were known instead, the surface temperature rise could be solved iteratively using:

$$q'' = \overline{h}(T_s - T_\infty)$$

But here, using the assumed surface temperature gives a first-order estimation of heat loss.

```
clc;
clear;
close all;
% Parameters
g = 9.81;
L = 0.5;            % Height [m]
W = 0.5;            % Width [m]
TW = 400;           % Plate surface temperature [K]
TINF = 300;         % Ambient temperature [K]
TF = (TW + TINF)/2;

% Air properties at ~TF
nu = 20.92e-6;
k = 30e-3;
Pr = 0.70;
beta = 1/TF;

% Grid resolution
nx = 200;
ny = 100;
```

```
x = linspace(0.01, L, nx);     % Height
y = linspace(0, W, ny);        % Width
[X, Y] = meshgrid(x, y);       % 2D grid

% Existing computations
Gr_x = (g * beta * (TW - TINF) .* X.^3) ./ (nu^2);
Ra_x = Gr_x .* Pr;
Nu_x = 0.508 * (Pr ./ (Pr + 0.952)).^(1/4) .* Ra_x.^(1/4);
h_x = Nu_x .* k ./ X;
q_x = h_x .* (TW - TINF);
delta_x = 3.93 * ((Pr + 0.952) ./ (Pr^2 .* Gr_x)).^(1/4) .* X;

% Adding temperature profile using exponential boundary layer form
% T(x,y) = T_inf + (T_w - T_inf)*exp(-y/delta(x))
T_profile = TINF + (TW - TINF) .* exp(-Y ./ delta_x);

%% Spatially Varying Surface Heat Flux
q0 = 300;  % Base heat flux [W/m^2]
q_input = q0 * (1 + 0.5 * sin(2 * pi * x / L));  % Imposed spatial heat flux
T_wx = q_input ./ h_x(1,:) + TINF;              % Resulting wall temperature
beta_x = 1 ./ ((T_wx + TINF) / 2);              % Local beta
Gr_x_new = (g .* beta_x .* (T_wx - TINF) .* x.^3) ./ (nu^2);
Ra_x_new = Gr_x_new .* Pr;
Nu_x_new = 0.508 * (Pr ./ (Pr + 0.952)).^(1/4) .* Ra_x_new.^(1/4);
h_x_new = Nu_x_new .* k ./ x;
q_conv = h_x_new .* (T_wx - TINF);
q_cond = k_cond * (T_wx - TINF) ./ plate_thickness;

% Plotting
n = 1;
% Line Plots
figure(n); n = n + 1;
plot(x, Gr_x(1,:), 'r', 'LineWidth', 2);
xlabel('x [m]'); ylabel('Gr_x'); title('Grashof Number vs. x'); grid on;

figure(n); n = n + 1;
plot(x, Ra_x(1,:), 'b', 'LineWidth', 2);
xlabel('x [m]'); ylabel('Ra_x'); title('Rayleigh Number vs. x'); grid on;

figure(n); n = n + 1;
plot(x, Nu_x(1,:), 'm', 'LineWidth', 2);
xlabel('x [m]'); ylabel('Nu_x'); title('Nusselt Number vs. x'); grid on;

figure(n); n = n + 1;
plot(x, h_x(1,:), 'k', 'LineWidth', 2);
xlabel('x [m]'); ylabel('h(x) [W/m^2K]'); title('Heat Transfer Coefficient vs. x'); grid
    on;

figure(n); n = n + 1;
plot(x, q_x(1,:), 'g', 'LineWidth', 2);
xlabel('x [m]'); ylabel('q''(x) [W/m^2]'); title('Heat Flux vs. x'); grid on;

figure(n); n = n + 1;
plot(x, delta_x(1,:) * 1000, 'c', 'LineWidth', 2);
xlabel('x [m]'); ylabel('\delta(x) [mm]'); title('Boundary Layer Thickness'); grid on;
% Conduction vs Convection Comparison
figure;
plot(x, q_conv, 'r', 'LineWidth', 2); hold on;
plot(x, q_cond, 'b--', 'LineWidth', 2);
xlabel('x [m]'); ylabel('Heat Flux [W/m^2]');
title('Convection vs. Conduction Heat Flux');
legend('q_{conv}', 'q_{cond}');
grid on;

% Plot: Imposed q" vs Resulting T_w(x)
figure;
plot(x, q_input, 'g', 'LineWidth', 2); hold on;
plot(x, T_wx, 'm', 'LineWidth', 2);
xlabel('x [m]'); ylabel('q" or T_w(x)');
title('Imposed Heat Flux and Resulting Wall Temperature');
legend('q_{input}(x)', 'T_w(x)');
grid on;
% Contour Plots
```

```
figure(n); n = n + 1;
contourf(X, Y, Gr_x, 30, 'LineColor','none');
colorbar; title('Contour of Grashof Number Gr_x');
xlabel('x [m]'); ylabel('y [m]');

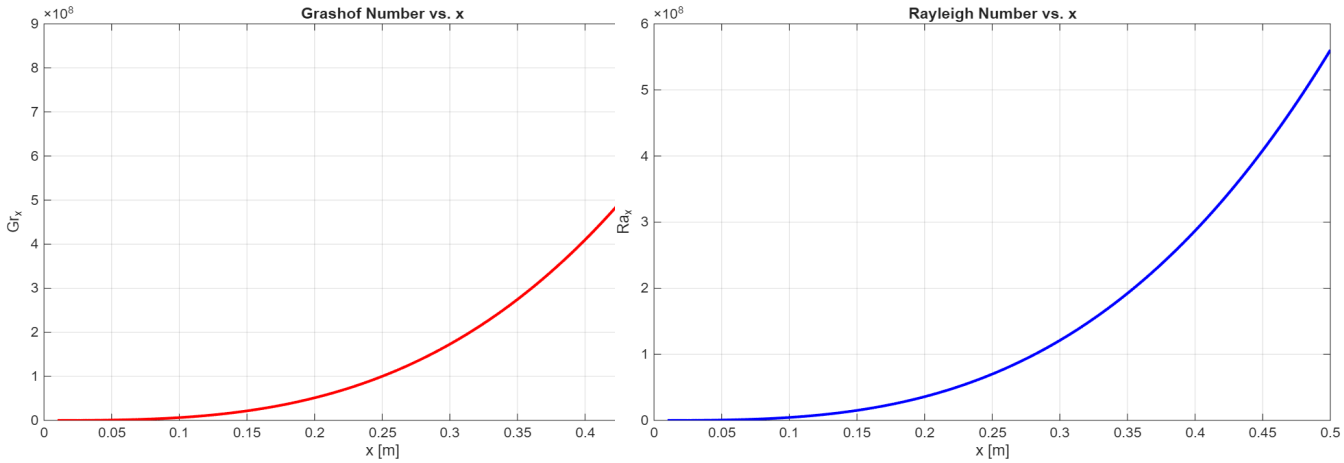
figure(n); n = n + 1;
contourf(X, Y, Ra_x, 30, 'LineColor','none');
colorbar; title('Contour of Rayleigh Number Ra_x');
xlabel('x [m]'); ylabel('y [m]');

figure(n); n = n + 1;
contourf(X, Y, Nu_x, 30, 'LineColor','none');
colorbar; title('Contour of Nusselt Number Nu_x');
xlabel('x [m]'); ylabel('y [m]');

figure(n); n = n + 1;
contourf(X, Y, h_x, 30, 'LineColor','none');
colorbar; title('Contour of Heat Transfer Coefficient h(x)');
xlabel('x [m]'); ylabel('y [m]');

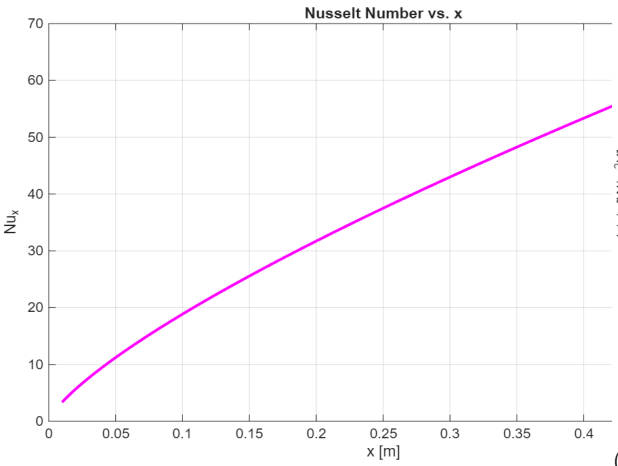
figure(n); n = n + 1;
contourf(X, Y, q_x, 30, 'LineColor','none');
colorbar; title('Contour of Heat Flux q''(x)');
xlabel('x [m]'); ylabel('y [m]');
% Temperature Contour
figure(n); n = n + 1;
contourf(X, Y, T_profile, 30, 'LineColor', 'none');
colorbar;
title('Dimensional Temperature Profile T(x,y) [K]');
xlabel('x [m]'); ylabel('y [m]');
%Temperature profiles at specific x locations
x_slices = [0.05, 0.1, 0.2, 0.4];  % horizontal positions
colors = ['r', 'b', 'g', 'm'];
figure(n); n = n + 1;
hold on;
for i = 1:length(x_slices)
    [~, idx] = min(abs(x - x_slices(i)));
    plot(y, T_profile(:, idx), colors(i), 'LineWidth', 2, 'DisplayName', ['x = ' num2str(
    x_slices(i)) ' m']);
end
hold off;
xlabel('y [m]');
ylabel('T(y) [K]');
legend('show');
title('Vertical Temperature Profiles at Selected x-Locations');
grid on;
```

Listing 1: MATLAB Code for PLotting Curves and Contours

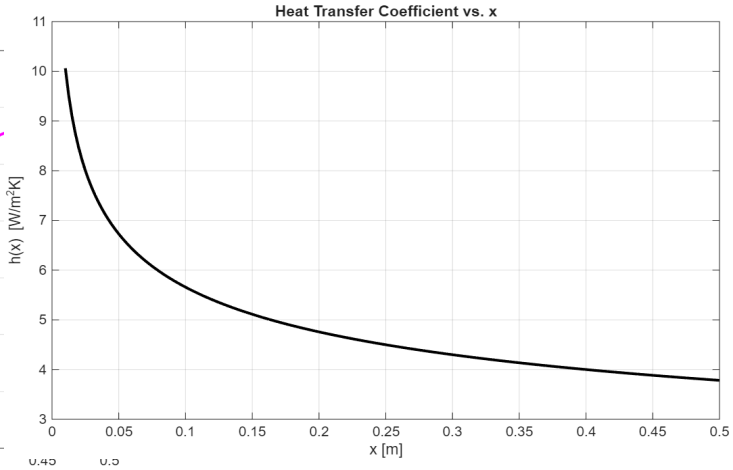


(a) Plot for Grashof Number vs. Horizontal Distance (b) Plot for Reyleigh Number vs. Horizontal Distance

Figure 1: Plots for Grashof Number and Reyleigh Number vs.Horizontal Distance

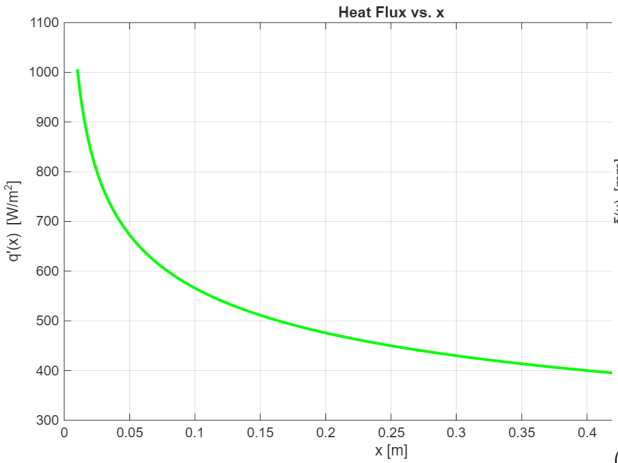


(a) Plot for Nusselt Number vs. Horizontal Distance

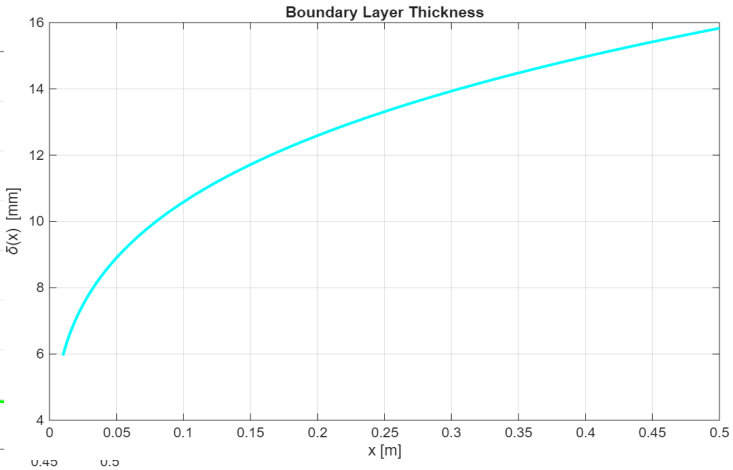


(b) Plot for Heat Transfer Coefficient vs. Horizontal Distance

Figure 2: Plots for Nusselt Number and Heat transfer coefficient vs.Horizontal Distance

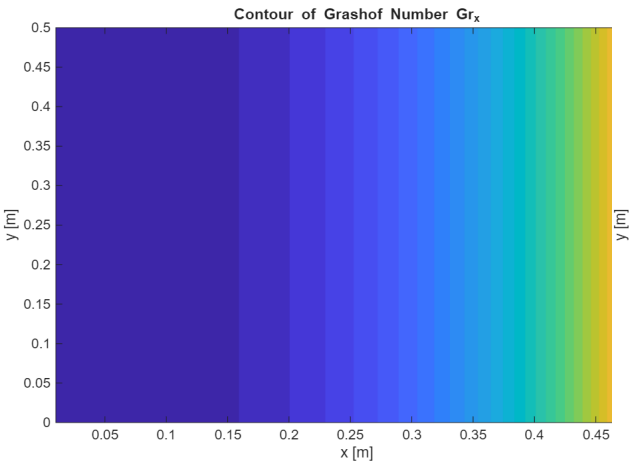


(a) Plot for Heat Flux vs. Horizontal Distance

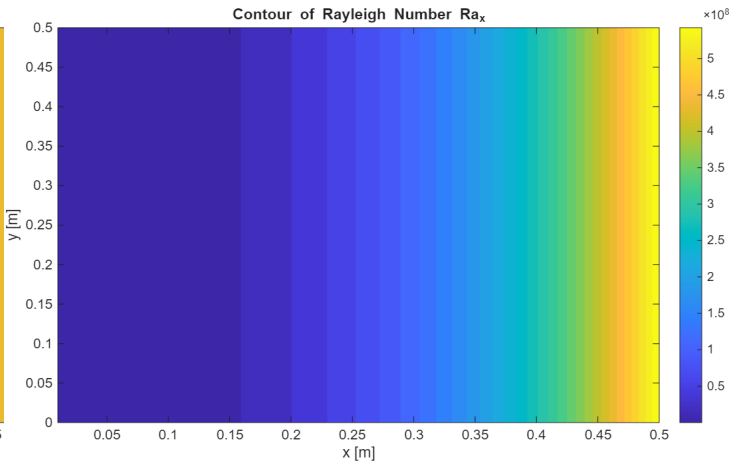


(b) Plot for Boundary Layer Thickness vs. Horizontal Distance

Figure 3: Plot for Heat Flux and Boundary Layer Thickness vs. Horizontal Distance

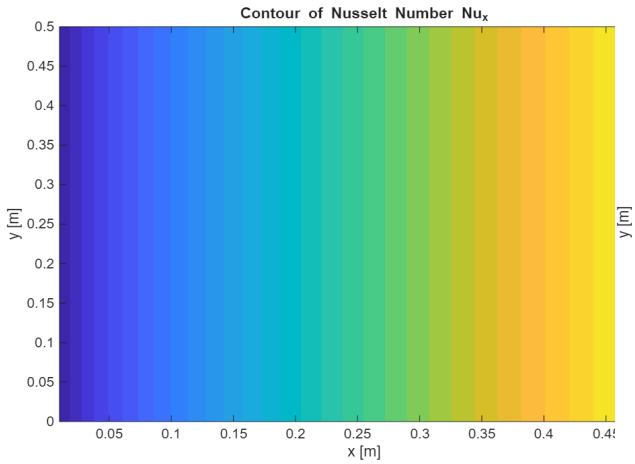


(a) Contour for Grashof Number

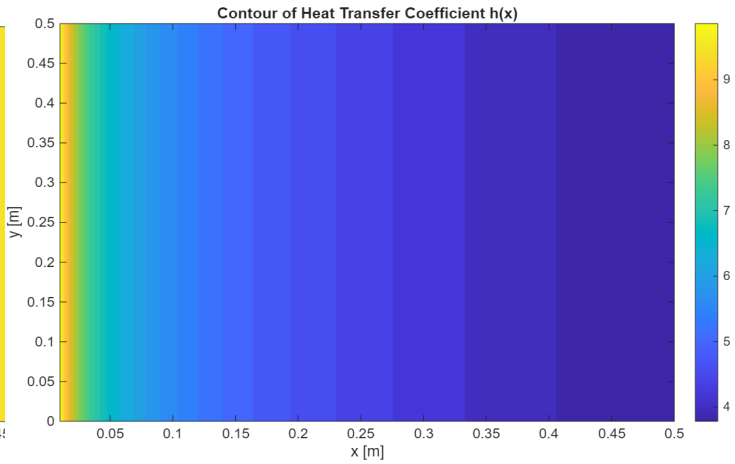


(b) Contour for Reyleigh Number

Figure 4: Contours for Grashof Number and Reyleigh Number

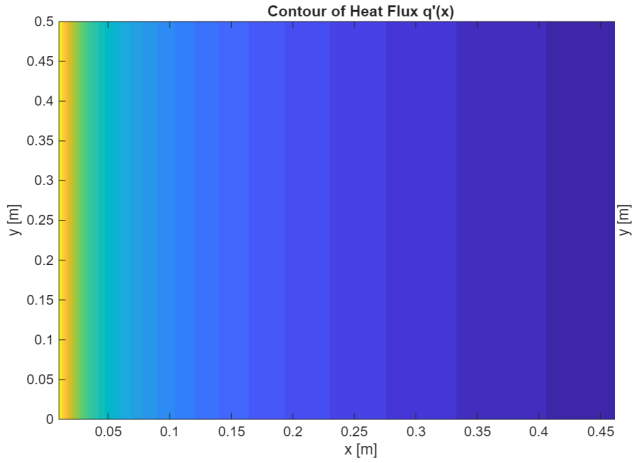


(a) Contour for Nusselt Number

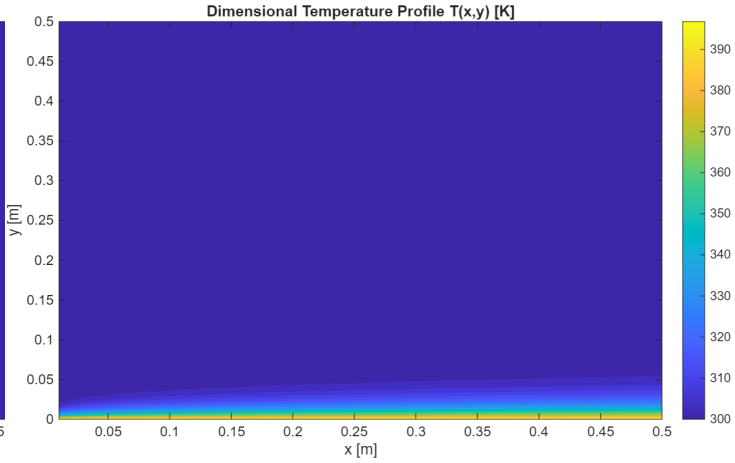


(b) Contour for Heat Transfer Coefficient

Figure 5: Contours for Nusselt Number and Heat Transfer Coefficient



(a) Contour for Heat Flux



(b) Contour for Dimensional Temperature Profile

Figure 6: Contours for Heat Flux and Dimensional Temperature Profiles

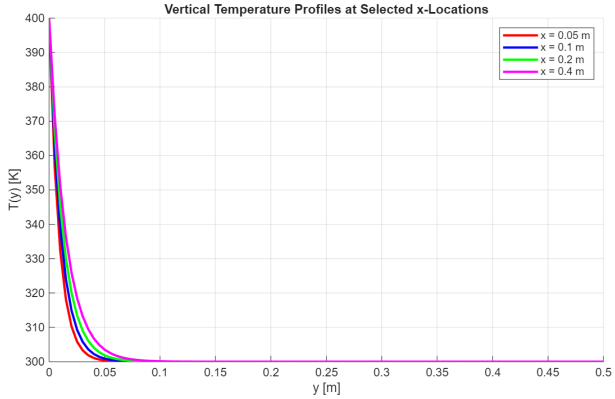


Figure 7: Plot for Vertical Temperature Profile at various Horizontal Points

Spatial Variation of Heat Flux and Comparison of Heat Transfer through Conduction and Convection

The heat flux can be varied spatially as:

$$\ddot{q}_{input}(x) = \ddot{q}_o(1 + 0.5sin(\frac{2\pi x}{L}))$$

The wall temperature can be determined through the following expression:

$$T_w(x) = \frac{q^{"}_{input}(x)}{h(x)} + T_\infty$$

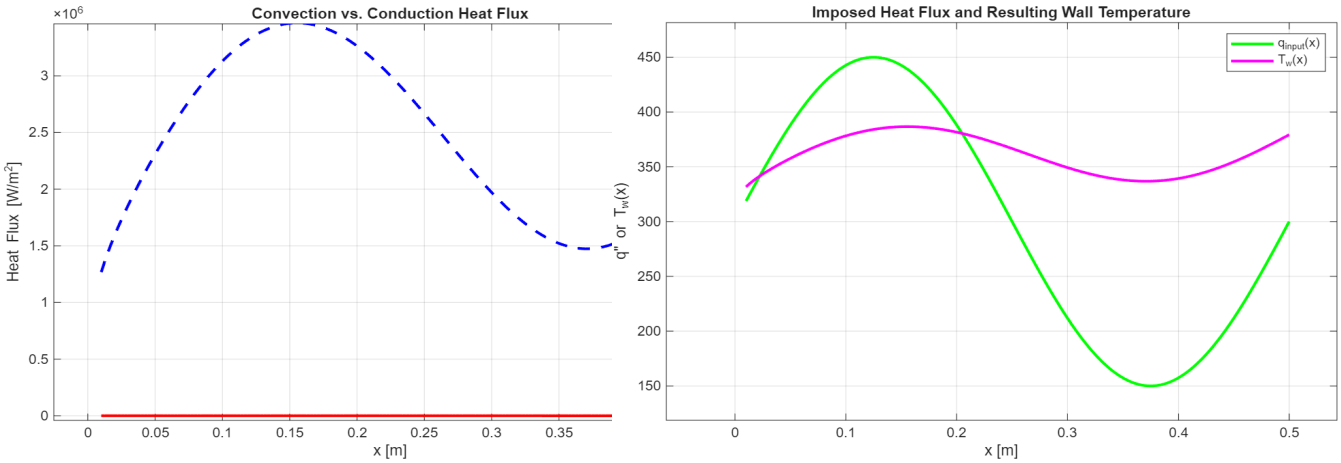
The heat transfer through conduction is given as:

$$q^{"}_{cond} = \frac{k\Delta T}{t}$$

The heat transfer through convection is given as:

$$q^{"}_{conv} = h(x).(T_w - T_\infty)$$

The consequent plots are as follows:



(a) Comparison between Heat Flux

(b) Plot for Imposed Heat Flux and Wall Temperature

Figure 8: Plots for Comparison between Conductive and Convective Heat Flux and Imposed Heat Flux and Wall Temperature

Conclusion

The approximate analysis of laminar free convection on a vertical plate using the von Kármán–Pohlhausen method has demonstrated high accuracy and alignment in predicting boundary layer behavior and heat transfer characteristics for constant wall temperature conditions. By employing assumed polynomial profiles and simplifying integral conservation equations under the Boussinesq approximation, this study achieved analytical expressions for thermal and hydrodynamic variables that align well with classical solutions. The derivation of the Nusselt number as a function of the Rayleigh and Prandtl numbers provided significant information regarding the influence of fluid properties on convective performance. The accompanying MATLAB simulation further enhanced understanding through visual analysis of parameter evolution along the plate. Overall, this work offers a computationally efficient and pedagogically valuable tool for researchers and educators involved in thermal-fluid sciences.

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