An Approximate Analysis of Laminar Free Convection on a Vertical Plate using von Karman-Pohlhausen Method: Constant Wall Temperature

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3rd June 2025

Abstract

This study presents a robust analytical formulation of laminar free convection over a vertical flat plate with constant wall temperature, using the von Kármán–Pohlhausen integral method. By assuming the polynomial profiles for velocity and temperature and making use of the Boussinesq approximation, together with the integral momentum and energy equation, expressions were derived and simplified into first-order differential equations. Through a power-law solution approach, expressions for the boundary layer thickness, velocity scale, local heat transfer coefficient, and Nusselt number were obtained in terms of the Rayleigh and Prandtl numbers. The analytical results were later verified using a practical example of air flowing over a heated vertical plate, which strongly aligned with analytical results, thus confirming both the accuracy and computational efficiency of the method. MATLAB visualizations of Grashof, Rayleigh, and Nusselt numbers, heat flux, and boundary layer development were devised which enabled reproducibility .

Keywords: Free convection; von Kármán–Pohlhausen method; vertical plate; boundary layer analysis; Nusselt number; MATLAB visualization

Consider the integral and momentum and energy equations by assuming velocity and temperature profiles. Consider the integral momentum equation as:

$$-\tau_w = \frac{1}{R}\frac{d}{dx}(R\int_0^Y \rho u^2 dy) - \frac{U_\infty}{R}\frac{d}{dx}(R\int_0^Y \rho u 2dy) + v_w\rho_w U_\infty + \int_0^Y (-\rho U_\infty \frac{dU_\infty}{dx})dy \tag{1}$$

The said procedure has been highlighted in [6]. By considering the body force term and adding it yo the momentum equation involves assuming $U_\infty = 0, v_w = 0$ and $R =$ constant, and along the added body force term, the integral term reduces to the following expression:

$$-\tau_w = \frac{d}{dx}(\int_0^Y \rho u^2 dy) + Y\frac{dp}{dx} + \int_0^Y \rho g dy \tag{2}$$

The pressure gradient term is given as:

$$\frac{dp}{dx} = -\rho_\infty g \tag{3}$$

By substitution of in results in:

$$\begin{aligned} -\tau_w &= \frac{d}{dx}\left(\int_0^Y \rho u^2\, dy\right) - Y\rho_\infty g + \int_0^Y \rho g\, dy \\ &= \frac{d}{dx}\left(\int_0^Y \rho u^2\, dy\right) - \int_0^Y \rho_\infty g\, dy + \int_0^Y \rho g\, dy \\ &= \frac{d}{dx}\left(\int_0^Y \rho u^2\, dy\right) - g\int_0^Y (\rho_\infty - \rho)\, dy. \end{aligned} \tag{4}$$

Using Boussinesq approximation, the density difference can be approximated as:

$$\rho_\infty - \rho \approx \beta\rho(T - T_\infty) \tag{5}$$

The substitution of in 4 results in:

$$-\tau_w = \frac{d}{dx}(\int_0^Y \rho u^2\, dy) - g\beta\rho\int_0^Y (T - T_\infty)\, dy \tag{6}$$

or

$$\rho\frac{d}{dx}(\int_0^\delta u^2\,dy) = -\mu(\frac{\partial u}{\partial y})_{y=0} + \rho g\beta\int_0^\delta (T-T_\infty)\,dy \tag{7}$$

The above equation validates both for laminar and turbulent flows. The velocity gradient about the edge of boundary layer tends to vanish, i.e $\partial u(x,\delta)/\partial y = 0$. This means that the shear force term vanishes. Considering the integral energy equation:

$$q_w^{"} = \frac{1}{R}\frac{d}{dx}[R\int_0^Y \rho u(h^* - h_\infty^*)\,dy] \tag{8}$$

The applicable energy equation can be written thus, while assuming that $h^* - h_\infty^* = c_p(T - T_\infty)$ and hence the Boussinseq approximation can be written as:

$$q_w^{"} = \rho c_p\frac{d}{dx}[\int_0^Y u(T-T_\infty)\,dy] \tag{9}$$

or

$$\rho c_p\frac{d}{dx}[\int_0^Y u(T-T_\infty)\,dy] = -k(\frac{\partial T}{\partial y})_{y=0} \tag{10}$$

It is generally assumed that the velocity and thermal boundary layers have same local thickness. The assumptions taken into account for deriving then integral formulation of energy equation are as follows:

- No changes in kinetic and potential energy
- There is negligible conduction in axial direction
- No viscous dissipation
- Constant fluidic properties

During the integral analysis, it is assumed that $\delta \approx \Delta$, which is justified through the results obtained with this assumption that agrees quite nicely with the experimental work of Touloukian et al. (1948). Sparrow and Gregg (1956) performed analysis for $\delta \neq \Delta$ and the results currently available are close to their work. The following approach requires less work and computational power and effort. For detailed analysis of integral analysis, can be referred to Kakac et al. (2014) and Ghiaasiaan (2018)

Constant Wall Temperature

Considering the constant wall temperature, the surface temperature of the interface is usually known. The heat transfer about the wall has to be determined, by assuming laminar boundary layer flow and third-degree velocity polynomial as:

$$u(x,y) = a_o(x) + a_1(x)y + a_2(x)y^2 + a_3(x)y^3 \tag{11}$$

The corresponding boundary conditions are:

$$u(x,0) = 0 \tag{12}$$

$$u(x,\delta) = 0 \tag{13}$$

$$\frac{\partial u(x,\delta)}{\partial y} = 0 \tag{14}$$

$$\frac{\partial^2 u(x,\delta)}{\partial y^2} = \frac{-\beta g}{\nu}(T_w - T_\infty) \tag{15}$$

The is obtained from the x-momentum equation by setting y=0. Using the very boundary conditions, the following coefficients are obtained:

$$a_o = 0$$
$$a_1 = \frac{\beta g(T_w - T_\infty)}{4\nu}\delta$$
$$a_2 = \frac{-\beta g(T_w - T_\infty)}{2\nu}$$
$$a_3 = \frac{-\beta g(T_w - T_\infty)}{4\nu}\frac{1}{\delta}$$

The substitution of a_o, a_1, a_2 and a_3 result in the following velocity profile:

$$u = \frac{\beta g(T_w - T_\infty)}{4\nu}\delta y[1 - 2(\frac{y}{\delta}) + (\frac{y}{\delta})^2] \tag{16}$$

It can be written as:

$$u = u_c(\frac{y}{\delta})[1 - \frac{y}{\delta}]^2 \tag{17}$$

$\delta(x)$ ad $u_c(x)$ are unknown and u_c is evaluated as:

$$u_c = [\frac{\beta g(T_w - T_\infty)}{4\nu}\delta^2] \tag{18}$$

The next step involves determining the temperature profile, for said purpose, a second-degree polynomial shall be used:

$$T(x,y) = b_o(x) + b_1(x)y + b_2(x)y^2 \tag{19}$$

The corresponding boundary conditions are:

$$T(x,0) = T_w \tag{20}$$

$$T(x,\Delta) = T_\infty \tag{21}$$

$$\frac{\partial T(x,\Delta)}{\partial y} = 0 \tag{22}$$

The unknown coefficients b_o, b_1 and b_2 are evaluated as:

$$b_o = T_w$$

$$b_1 = \frac{-2(T_w - T_\infty)}{\Delta}$$

$$a_2 = \frac{(T_w - T_\infty)}{\Delta^2}$$

The substitution of b_0, b_1 and b_2 in results in the following expression:

$$T(x,y) = T_\infty + (T_w - T_\infty)[1 - \frac{y}{\Delta}]^2 \tag{23}$$

Since $\delta \approx \Delta$ thus:

$$T(x,y) = T_\infty + (T_w - T_\infty)[1 - \frac{y}{\delta}]^2 \tag{24}$$

The heat transfer coefficient h(x) is defined as:

$$h(x) = \frac{q_w^{"}}{(T_w - T_\infty)} = \frac{-k(\frac{\partial T}{\partial y})_{y=0}}{(T_w - T_\infty)} = \frac{2k}{\Delta(x)} \tag{25}$$

Since $\delta \approx \Delta$ thus:

$$h = \frac{2k}{\delta(x)} \tag{26}$$

We only need to determine the velocity boundary layer thickness $\delta(x)$. To determine $\delta(x)$, we substitute velocity and temperature profiles into the integral form of the momentum equation:)

$$-\nu\frac{u_c}{\delta} + \mathrm{g}\beta(\mathrm{T}_w - \mathrm{T}_\infty)\int_0^\delta \left[1 - \frac{y}{\delta}\right]^2 \mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{u_c^2}{\delta^2}\int_0^\delta y^2\left[1 - \frac{y}{\delta}\right]^4 \mathrm{d}y\right\}. \tag{27}$$

Evaluating the integrals and rearranging them yield

$$\frac{1}{105}\frac{\mathrm{d}}{\mathrm{d}x}\left[u_c^2\delta\right] = \frac{\beta\,\mathrm{g}\,(\mathrm{T}_w - \mathrm{T}_\infty)\,\delta}{3} - \nu\frac{u_c}{\delta}. \tag{28}$$

Next, we substitute velocity and temperature profiles into integral form of the energy equation. This will give us

$$2\alpha(\mathrm{T}_w - \mathrm{T}_\infty)\frac{1}{\delta} = (\mathrm{T}_w - \mathrm{T}_\infty)\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{u_c}{\delta}\int_0^\delta y\left[1 - \frac{y}{\delta}\right]^4 \mathrm{d}y\right\}. \tag{29}$$

Evaluating the integral and rearranging the result yields

$$\frac{1}{60}\frac{\mathrm{d}}{\mathrm{d}x}(u_c\,\delta) = \frac{\alpha}{\delta}. \tag{30}$$

We have two first-order differential equations and two dependent variables $u_c(x)$ and $\delta(x)$. We now assume a solution of the form given below

$$u_c(x) = A\,x^{\mathrm{m}} \tag{31}$$

$$\delta(x) = \mathrm{B}\,x^{\mathrm{n}} \tag{32}$$

where A, B, m, and n are constants. The process of substitution results in:

$$\frac{2\mathrm{m}+\mathrm{n}}{105}\mathrm{A}^2\,\mathrm{B}x^{2\mathrm{m}+\mathrm{n}-1} = \frac{1}{3}\beta\mathrm{g}\,(\mathrm{T}_w - \mathrm{T}_\infty)\mathrm{B}\,x^{\mathrm{n}} - \frac{\mathrm{A}}{\mathrm{B}}\nu\,x^{\mathrm{m}-\mathrm{n}} \tag{33}$$

$$\frac{\mathrm{m}+\mathrm{n}}{60}\,\mathrm{AB}x^{\mathrm{m}+\mathrm{n}-1} = \frac{\alpha}{\mathrm{B}}x^{-\mathrm{n}}. \tag{34}$$

The above equations are satisfied if the exponents of x in each equation are identical. Thus,

$$2\mathrm{m}+\mathrm{n}-1 = \mathrm{n} = \mathrm{m}-\mathrm{n} \tag{35}$$

$$\mathrm{m}+\mathrm{n}-1 = -\mathrm{n}. \tag{36}$$

Solving these equations yields

$$\mathrm{m} = \frac{1}{2}, \quad \mathrm{n} = \frac{1}{4}. \tag{37}$$

Using these values of m and n, after simplification, we obtain the following equations in terms of A and B:

$$\frac{1}{84}\mathrm{A}^2\,\mathrm{B} = \frac{1}{3}\beta\mathrm{g}(\mathrm{T_w} - \mathrm{T}_\infty)\mathrm{B} - \frac{\mathrm{A}}{\mathrm{B}}\nu \tag{38}$$

$$\frac{1}{80}\mathrm{AB} = \frac{\alpha}{\mathrm{B}}. \tag{39}$$

Solving these equations for A and B, we get

$$\mathrm{A} = 5.17\nu\Big(\frac{20}{21}+\mathrm{Pr}\Big)^{-1/2}\left[\frac{\mathrm{g}\,\beta\,(\mathrm{T_w}-\mathrm{T}_\infty)}{\nu^2}\right]^{1/2} \tag{40}$$

$$\mathrm{B} = 3.93\,\mathrm{Pr}^{-1/2}\Big(\frac{20}{21}+\mathrm{Pr}\Big)^{-1/4}\left[\frac{\mathrm{g}\,\beta(\mathrm{T_w}-\mathrm{T}_\infty)}{\nu^2}\right]^{-1/4}. \tag{41}$$

Substituting these into the original equations, introducing the definition of Grashof number Gr_x, and after some algebra, we get

$$\mathrm{u_c} = 5.17\nu\Big(\mathrm{Pr}+\frac{20}{21}\Big)^{-1/2}\frac{\sqrt{\mathrm{Gr}_x}}{x} \tag{42}$$

$$\frac{\delta}{x} = 3.93\mathrm{Pr}^{-1/2}\Big(\mathrm{Pr}+\frac{20}{21}\Big)^{1/4}\mathrm{Gr}_x^{-1/4} \tag{43}$$

$$\frac{\delta}{x} = 3.93\Big(1+\frac{20}{21}\frac{1}{\mathrm{Pr}}\Big)^{1/4}\mathrm{Ra}_x^{-1/4}. \tag{44}$$

The velocity boundary layer thickness can also be expressed in the following form:

$$\frac{\delta}{x} = 3.93\left[\frac{\mathrm{Pr}+0.952}{\mathrm{Pr}^2\,\mathrm{Gr}_x}\right]^{1/4}. \tag{45}$$

The local heat transfer coefficient $h(x)$ is

$$h(x) = \frac{2k}{\delta(x)} = \frac{k}{\chi}\frac{0.508\mathrm{Pr}^{-1/2}\mathrm{Gr}_L^{1/4}}{(0.952+\mathrm{Pr})^{1/4}} \tag{46}$$

and the local Nusselt number Nu_x is

$$\mathrm{Nu}_x = \frac{0.508\ \mathrm{Pr}^{-1/2}\ \mathrm{Gr}_L^{1/4}}{[\mathrm{Pr}+0.952]^{1/4}} \tag{47}$$

or in terms of the Rayleigh number Ra_x

$$\mathrm{Nu}_x = \frac{0.508\ \mathrm{Pr}^{-1/4}}{[0.952+\mathrm{Pr}]^{1/4}}\mathrm{Ra}_x^{1/4} \tag{48}$$

or

$$\mathrm{Nu}_x = 0.508\left(\frac{\mathrm{Pr}}{0.952+\mathrm{Pr}}\right)^{1/4}\mathrm{Ra}_x^{1/4} \tag{49}$$

where $\mathrm{Ra}_x = \frac{g\beta(T_w-T_\infty)x^3}{\nu\alpha}$ is the Rayleigh number.

The average heat transfer coefficient $\overline{h}$ is

$$\overline{h} = \frac{1}{L}\int_0^L h(x)\,\mathrm{d}x = 0.508\mathrm{Pr}^{-1/2}(0.952+\mathrm{Pr})^{-1/4}\left[\frac{g\beta\,(T_w - T_\infty)}{\nu^2}\right]^{1/4} k\frac{4}{3}\frac{1}{L^{1/4}} \tag{50}$$

and the average Nusselt number in terms of the Rayleigh number Ra_L

$$\overline{\mathrm{Nu}_L} = 0.68\left[\frac{\mathrm{Pr}}{0.952+\mathrm{Pr}}\right]^{1/4}\mathrm{Ra}_L^{1/4} \tag{51}$$

where $\mathrm{Ra}_L = \frac{g\beta(T_w - T_\infty)L^3}{\nu\alpha}$ is the Rayleigh number based on the plate length L.

Practical Example

The surface temperature of a square plate is 400 K. The one side of the plate is 0.5m, and the plate is in vertical position in quiescent air at 1 atm and 300 K. Determine:
(a) Determine the velocity boundary layer thickness at the trailing edge.
(b) Determine the total heat loss from the plate. Onur (2023)

Solution

The film temperature can be determined as:

$$T_f = \frac{T_w + T_\infty}{2} = \frac{400+300}{2}$$

$$T_f = 350\ K$$

Referring to the literature for air properties which are as follows:

$$\nu = 20.92\times10^{-6}m^2s^{-1}$$

$$k = 30\times10^{-3}W/mK$$

$$Pr = 0.700$$

The coefficient of thermal expansion is given as:

$$\beta = \frac{1}{T_f} = \frac{1}{350}$$

$$\beta = 0.00285K^{-1}$$

The Grashof number at the leading edge can be calculated as:

$$Gr_x = \frac{\beta g(T_w - T_\infty)x^3}{\nu^2} = \frac{9.81\times0.00285\times(400-300)\times(0.5)^3}{(20.92\times10^{-6})^2}$$

$$Gr_x = 8\times10^8$$

$$\delta = 3.93\left[\frac{\mathrm{Pr}+0.952}{\mathrm{Pr}^2\,\mathrm{Gr}_x}\right]^{1/4}x$$

$$\delta = 3.93[\frac{0.7+0.952}{0.7^2\times8\times10^8}]^{1/4}(0.5)$$

$$\delta = 0.01582\ m$$

The total heat transfer can be calculated from the plate as:

$$Ra_L = Gr_L Pr = 8\times10^8\times0.7$$

$$Ra_L = 5.6\times10^8$$

The average Nusselt number $\overline{Nu_L}$ can be calculated as:

$$\overline{\mathrm{Nu}_L} = 0.68\left[\frac{\mathrm{Pr}}{0.952+\mathrm{Pr}}\right]^{1/4}\mathrm{Ra}_L^{1/4}$$

$$\overline{\mathrm{Nu}_L} = 0.68\left[\frac{0.7}{0.952+0.7}\right]^{1/4}(5.6\times10^8)^{1/4}$$

The average heat transfer coefficient $\overline{h}$ can be calculated as:

$$\overline{h} = \frac{k}{L}\overline{Nu_L} = (\frac{30\times10^{-3}}{0.5})(84.41)$$

$$\overline{h} = 5.06\ W/m^2K$$

The total heat transfer can be evaluated as:

$$q = 2\times\overline{h}\times A\times(T - T_\infty) = 2\times(5.06)\times(0.5\times0.5)\times(400-300)$$

$$q = 253.26\ W$$

```
      clc;
clear;

%Parameters
g = 9.81;
L = 0.5;            % Height [m]
W = 0.5;            % Width [m]
TW = 400;           % Plate surface temperature [K]
TINF = 300;         % Ambient temperature [K]
TF = (TW + TINF)/2;

% Air properties at ~TF
nu = 20.92e-6;
k = 30e-3;
Pr = 0.70;
beta = 1/TF;
% Grid resolution
nx = 200;
ny = 100;

x = linspace(0.01, L, nx);    % Height
y = linspace(0, W, ny);       % Width
[X, Y] = meshgrid(x, y);      % 2D grid
Gr_x = (g * beta * (TW - TINF) .* X.^3) ./ (nu^2);
Ra_x = Gr_x .* Pr;
Nu_x = 0.508 * (Pr ./ (Pr + 0.952)).^(1/4) .* Ra_x.^(1/4);
h_x = Nu_x .* k ./ X;
q_x = h_x .* (TW - TINF);
delta_x = 3.93 * ((Pr + 0.952) ./ (Pr^2 .* Gr_x)).^(1/4) .* X;

%Line Plots
figure; plot(x, Gr_x(1,:), 'r', 'LineWidth', 2);
xlabel('x [m]'); ylabel('Gr_x'); title('Grashof Number vs. x'); grid on;
figure; plot(x, Ra_x(1,:), 'b', 'LineWidth', 2);
xlabel('x [m]'); ylabel('Ra_x'); title('Rayleigh Number vs. x'); grid on;
figure; plot(x, Nu_x(1,:), 'm', 'LineWidth', 2);
xlabel('x [m]'); ylabel('Nu_x'); title('Nusselt Number vs. x'); grid on;
figure; plot(x, h_x(1,:), 'k', 'LineWidth', 2);
xlabel('x [m]'); ylabel('h(x) [W/m^2K]'); title('Heat Transfer Coefficient vs. x'); grid
    on;
figure; plot(x, q_x(1,:), 'g', 'LineWidth', 2);
xlabel('x [m]'); ylabel('q''(x) [W/m^2]'); title('Heat Flux vs. x'); grid on;
figure; plot(x, delta_x(1,:) * 1000, 'c', 'LineWidth', 2);
xlabel('x [m]'); ylabel('\delta(x) [mm]'); title('Boundary Layer Thickness'); grid on;

%Contour Plots

figure;
contourf(X, Y, Gr_x, 30, 'LineColor','none');
colorbar; title('Contour of Grashof Number Gr_x');
xlabel('x [m]'); ylabel('y [m]');

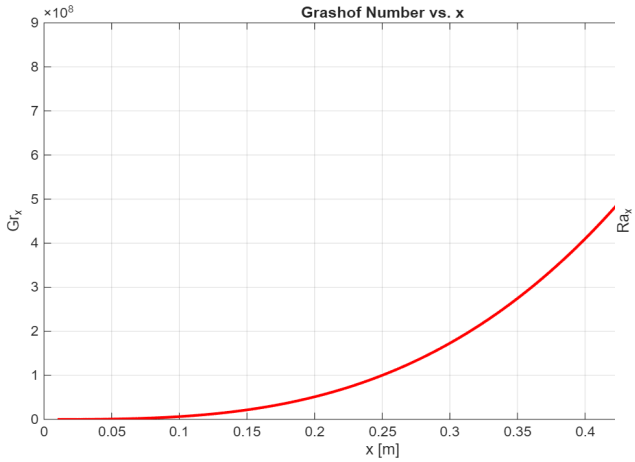
figure;
contourf(X, Y, Ra_x, 30, 'LineColor','none');
colorbar; title('Contour of Rayleigh Number Ra_x');
xlabel('x [m]'); ylabel('y [m]');

figure;
contourf(X, Y, Nu_x, 30, 'LineColor','none');
colorbar; title('Contour of Nusselt Number Nu_x');
xlabel('x [m]'); ylabel('y [m]');

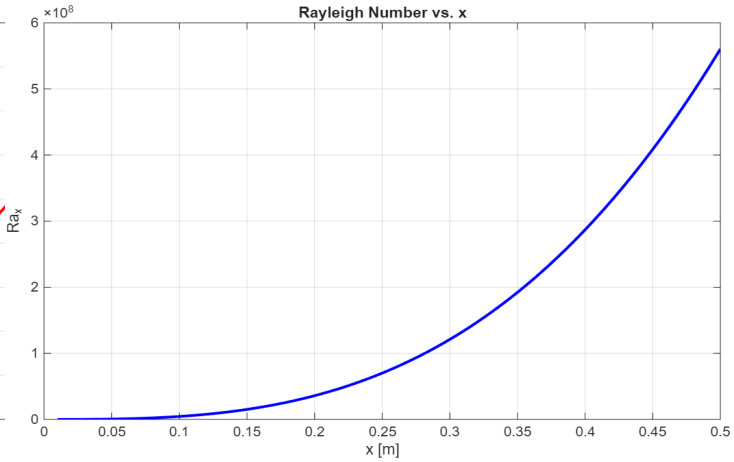
figure;
contourf(X, Y, h_x, 30, 'LineColor','none');
colorbar; title('Contour of Heat Transfer Coefficient h(x)');
xlabel('x [m]'); ylabel('y [m]');

figure;
contourf(X, Y, q_x, 30, 'LineColor','none');
colorbar; title('Contour of Heat Flux q''(x)');
xlabel('x [m]'); ylabel('y [m]');
```

Listing 1: MATLAB Code for PLotting Curves and Contours

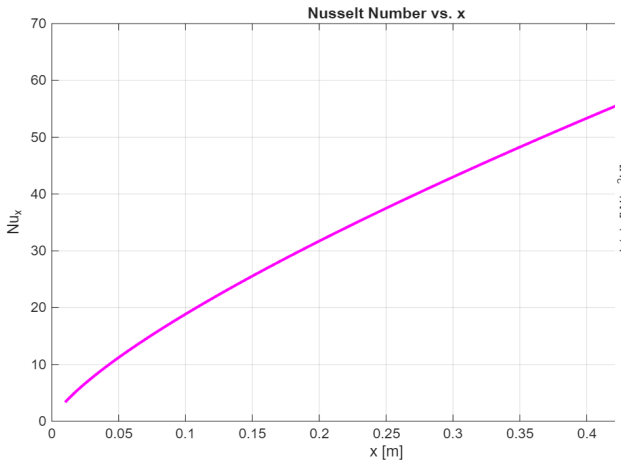


(a) Plot for Grashof Number vs. Horizontal Distance

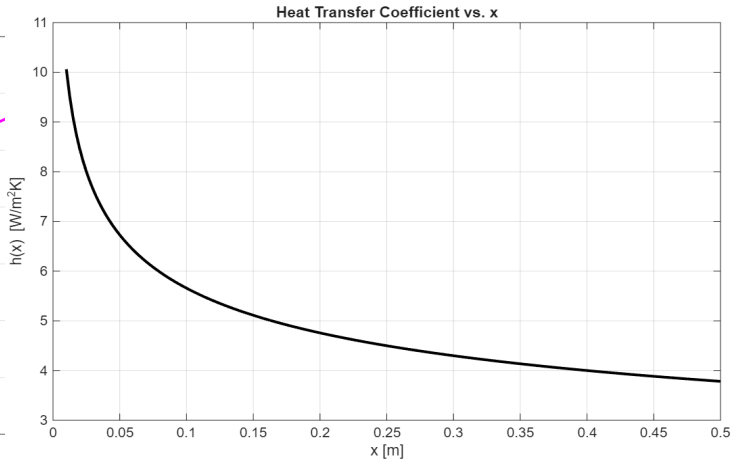


(b) Plot for Reyleigh Number vs. Horizontal Distance

Figure 1: Plots for Grashof Number and Reyleigh Number vs.Horizontal Distance

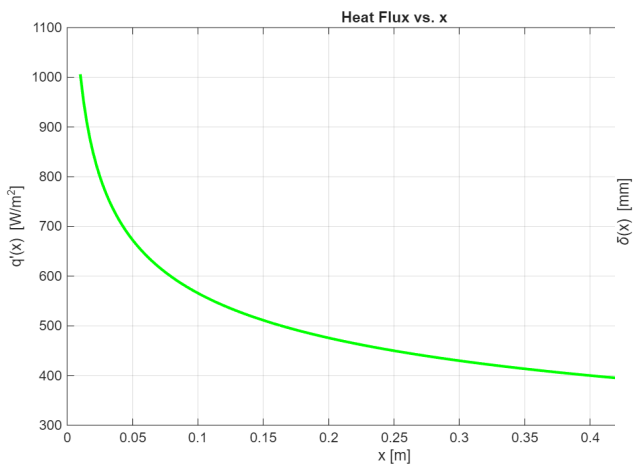


(a) Plot for Nusselt Number vs. Horizontal Distance

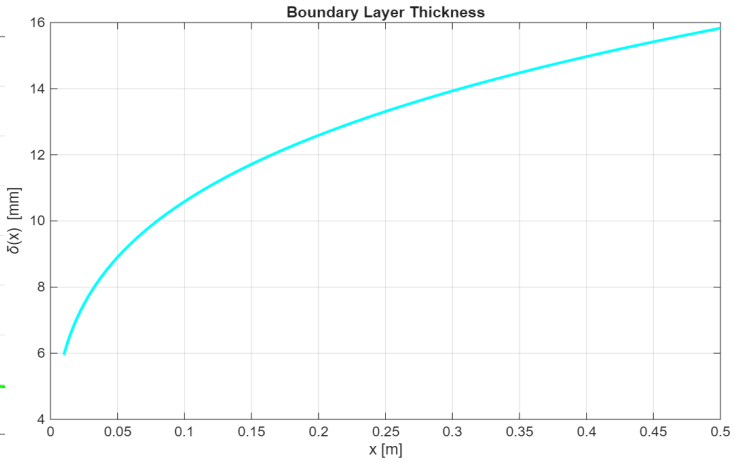


(b) Plot for Heat Transfer Coefficient vs. Horizontal Distance

Figure 2: Plots for Nusselt Number and Heat transfer coefficient vs.Horizontal Distance

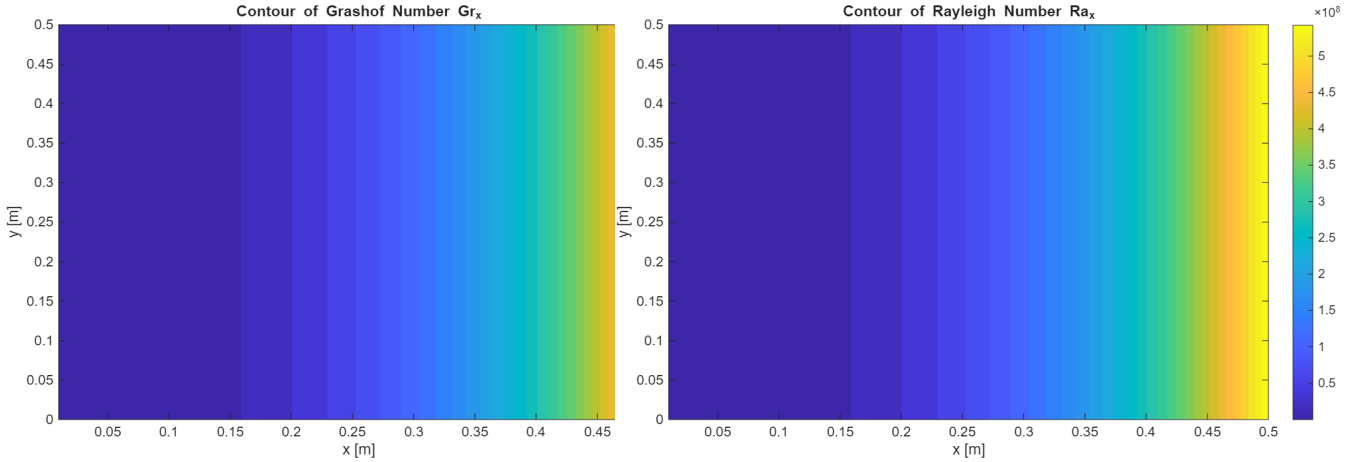


(a) Plot for Heat Flux vs. Horizontal Distance



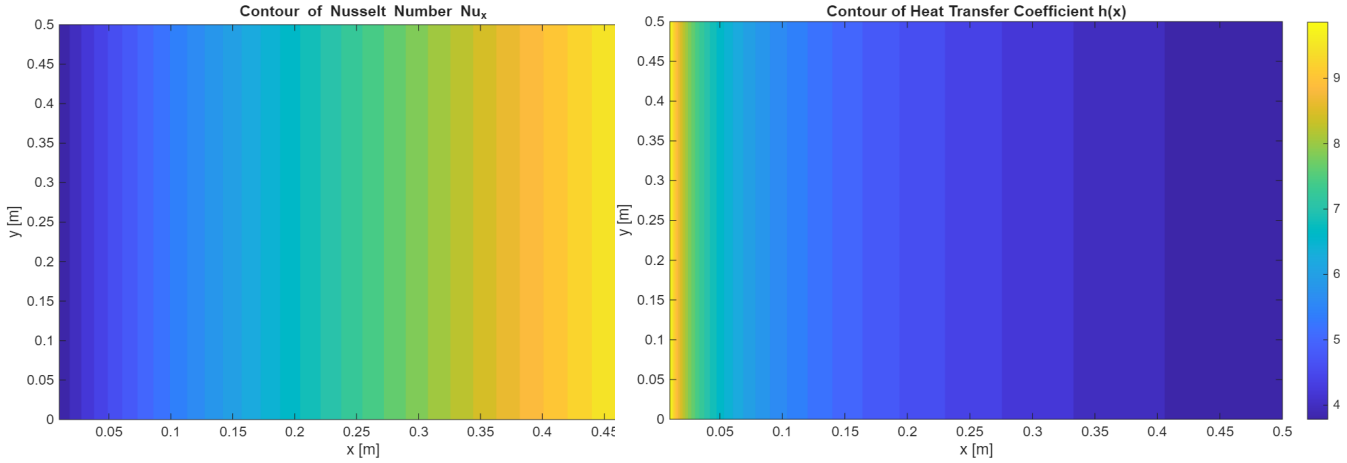
(b) Plot for Boundary Layer Thickness vs. Horizontal Distance

Figure 3: Plot for Heat Flux and Boundary Layer Thickness vs. Horizontal Distance



(a) Contour for Grashof Number (b) Contour for Reyleigh Number

Figure 4: Contours for Grashof Number and Reyleigh Number



(a) Contour for Nusselt Number (b) Contour for Heat Transfer Coefficient

Figure 5: Contours for Nusselt Number and Heat Transfer Coefficient

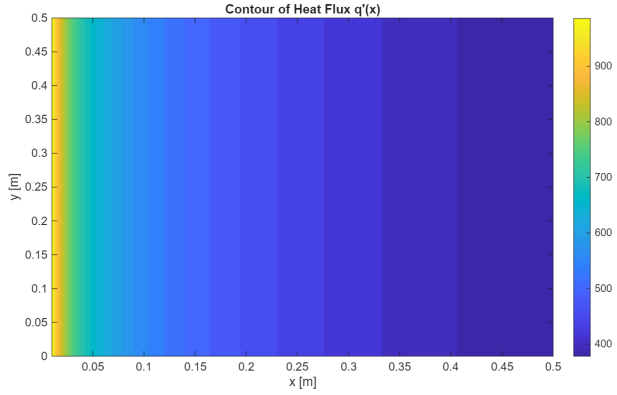


Figure 6: Contour for Heat Flux

Conclusion

The approximate analysis of laminar free convection on a vertical plate using the von Kármán–Pohlhausen method has demonstrated high accuracy and alignment in predicting boundary layer behavior and heat transfer characteristics for constant wall temperature conditions. By employing assumed polynomial profiles and simplifying integral conservation equations under the Boussinesq approximation, this study achieved analytical expressions for ther-

mal and hydrodynamic variables that align well with classical solutions. The derivation of the Nusselt number as a function of the Rayleigh and Prandtl numbers provided significant information regarding the influence of fluid properties on convective performance. The accompanying MATLAB simulation further enhanced understanding through visual analysis of parameter evolution along the plate. Overall, this work offers a computationally efficient and pedagogically valuable tool for researchers and educators involved in thermal-fluid sciences.

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