AN ANALYTICAL APPROACH TO HEAT TRANSFER FOR SLUG FLOW IN CIRCULAR TUBE SUBJECTED TO CONSTANT WALL TEMPERATURE

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Abstract. This document presents an analytical investigation of heat transfer characteristics in a laminar slug flow through a circular tube subjected to constant wall temperature. The study emphasizes on the undeveloped hydrodynamic boundary layer about the entrance region of pipe, where the velocity is approximated to be uniform. For low-Prandtl number fluids, $Pr < 1$, thermal diffusivity dominates over momentum diffusivity, thus justifying the neglect of axial conduction and viscous dissipation, The energy equation is solved using variable separation method, resulting in dimensionless temperature distribution expressed in terms of Bessel function of first kind. The resulting eignevalue problem is solved to obtain characteristic roots λ_n, resulting in closed-form expression for temperature field and mean fluid temperature, The local and average Nusselt numbers are derived as function of Graetz and Peclet numbers, demonstrating the convergence towards an asymptotic value $Nu_D = 5.7831$, for fully developed laminar flow. The analysis demonstrates a rigorous mathematical framework for predicting thermal development in slug flow under constant wall temperature conditions, providing valuable information for microchannel cooling, gas-liquid transport systems and heat exchanger optimization

Keywords. hydrodynamic boundary layer, Prandtl number, thermal diffusivity, Bessel function, Graetz number, Peclet number

1. Circular pipe: Slug Flow Heat Transfer in the Entrance Region

Slug flow is defined as fluid flow in the pipe regime where small air bubble transform to large air bubbles, filling the entire pipe cross-section, with liquid slugs containing smaller entrained gas bubbles present between them. Slug flows are typically observed at low values of Prandtl number.
Considering the short unheated entrance length of tube, with small value of Prandtl number (Pr$<$ 1), where $\delta/\Delta \to 1$, implies that the momentum boundary layer can be ignored from analysis in comparison with the thermal boundary layer. The rate of development of temperature profile is faster than that of velocity profile. With this, the axial velocity can be assumed to be constant and it can be approximated as

$$v_z = V$$

where V is the average fluid velocity. The said assumption is valid only for small value of entrance length and it is not valid for large distances downstream since

the velocity profile shall ultimately develop.

One may consider the two-dimensional,incompressible, steady laminar flow of

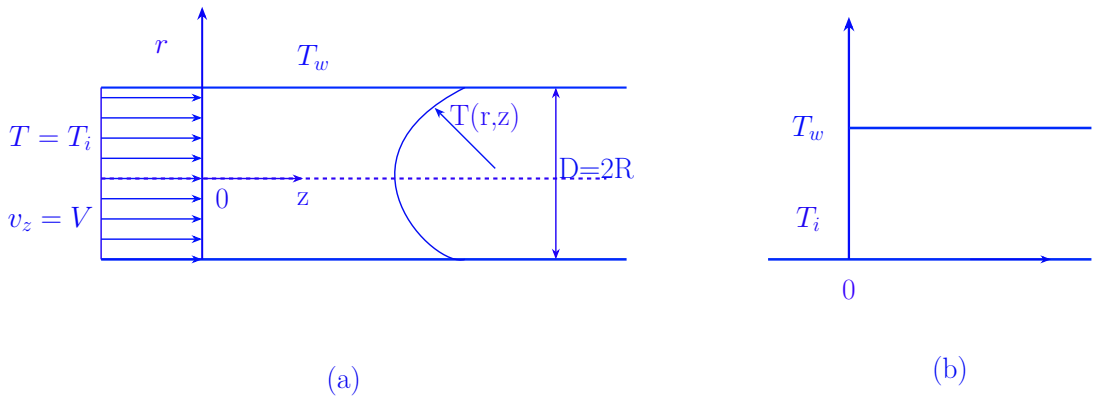


FIGURE 1. (a) Slug flow in circular tube (b) Constant wall temperature boundary condition

a low-Prandtl number fluid flowing in the entrance region of the pipe as seen in **??** The fluid properties are assumed to be constant, with negligible viscous dissipation and no internal energy generation. The pipe radius is denoted as R, where the fluid enters the pipe with an initial temperature of T_i and uniform velocity V. The flow is for low-Prandtl number, where $\delta/\Delta \to 1$, the rate of development of temperature profile is faster than that of velocity profile, implying greater effect of diffusivity. With this, the axial velocity can be assumed to be constant and it can be approximated as $v_z = V$. The coordinate system is located at the duct inlet where the heating begins progressively. Heat flow gradually transforms into slug flow from the thermal pipe wall, which is maintained at constant temperature T_w. The constant area is heated across its entire length, hence it can be construed that the thermal boundary layer development begins at the entrance of the duct and develops along its entire length. Axial conduction and viscous dissipation are ignored from the analysis, with no energy generation. The temperature distribution and the empirical relation for the Nusselt number is to be determined.

The energy equation at the entrance region is to be calculated as

$$V\frac{\partial T}{\partial z} = \alpha\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] \tag{1.1}$$

The corresponding boundary conditions are

$$T(0, r) = T_i \tag{1.2}$$

$$T(z, R) = T_w \tag{1.3}$$

$$\frac{\partial T(z, 0)}{\partial r} = 0 \tag{1.4}$$

The dimensionless variables for broader analysis can be defined here as

$$\theta = \frac{T - T_w}{T_i - T_w} \tag{1.5}$$

$$\eta = \frac{r}{R} \tag{1.6}$$

$$z^+ = \frac{z}{z_o} \tag{1.7}$$

z_o is an unknown parameter and hence shall be determined as follows

$$\frac{\partial T}{\partial z} = (T_i - T_w)\frac{\partial T}{\partial z} = (T_i - T_w)\frac{\partial \theta}{\partial z^+}\frac{\partial z^+}{\partial z} = \frac{T_i - T_w}{z_o}\frac{\partial \theta}{\partial z^+}$$

$$\frac{\partial T}{\partial r} = (T_i - T_w)\frac{\partial \theta}{\partial r} = (T_i - T_w)\frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{T_i - T_w}{R}\frac{\partial \theta}{\partial \eta}$$

$$\frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{r}\frac{(T_i - T_w)}{R}\frac{\partial \theta}{\partial \eta}$$

$$r = \eta R$$

$$\frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\eta}\frac{(T_i - T_w)}{R^2}\frac{\partial \theta}{\partial \eta} = \frac{(T_i - T_w)}{R^2}\frac{1}{\eta}\frac{\partial \theta}{\partial \eta}$$

$$\frac{\partial^2 T}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{(T_i - T_w)}{R}\frac{\partial \theta}{\partial \eta}\right] = \frac{\partial}{\partial \eta}\left[\frac{(T_i - T_w)}{R}\frac{\partial \theta}{\partial \eta}\right]\frac{\partial \eta}{\partial r} = \left(\frac{(T_i - T_w)}{R^2}\right)\frac{\partial^2 \theta}{\partial \eta^2}$$

Substituting all the above in 1.1, the following expression can be obtained

$$V = \left(\frac{T_i - T_w}{z_o}\right)\frac{\partial \theta}{\partial z^+} = \alpha\left[\frac{(T_i - T_w)}{R^2}\frac{1}{\eta}\frac{\partial \theta}{\partial \eta} + \left(\frac{T_i - T_w}{R^2}\right)\frac{\partial^2 \theta}{\partial \eta^2}\right]$$

The unknown parameter z_o is now evaluated as

$$\frac{V}{z_o} = \frac{\alpha}{R^2} \implies z_o = \frac{VR^2}{\alpha}$$

The axial dimensionless distance can now be expressed as

$$z^+ = \frac{z\alpha}{R^2 V} = \frac{2(z/R)}{RePr} = \frac{4(z/D)}{Pe} = \frac{4}{Gz}$$

where $Gz = \left(\frac{D}{z}\right) RePr = \frac{D^2 V}{\alpha z} = \frac{4}{z^+}$ is the Graetz number, and Pe is the Peclet number and defined as $Pe = Re_D Pr$. The dimensionless temperature distribution is $\theta = \theta(z^+, \eta)$. An equation, therefore, has to be solved in dimensionless form which ultimately shall become a boundary condition

$$\frac{\partial \theta}{\partial z^+} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\partial \eta}\right) \tag{1.8}$$

$$\theta(0, \eta) = 1 \tag{1.9}$$

$$\theta(z^+, 1) = 0 \tag{1.10}$$

$$\frac{\partial \theta}{\partial \eta}(z^+, 0) = 0 \tag{1.11}$$

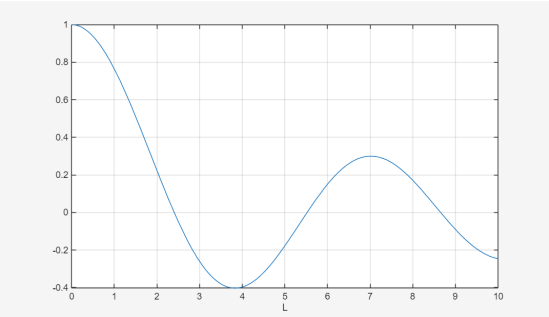


FIGURE 2. MATLAB interpretation of Maple solution for Bessel Function of J-Type

The solution for the following PDE has to be assumed to be in the following form

$$\theta = Z(z^+)R(\eta) \tag{1.12}$$

Plugging Equation 1.12 in Equation 1.8 results in two different ODEs

$$\frac{dZ}{dz^+} + \lambda^2 Z = 0 \implies Z(z^+) = e^{-\lambda^2 z^+} \tag{1.13}$$

$$\frac{d^2\Re}{d\eta^2} + \frac{1}{\eta}\frac{d\Re}{d\eta} + \lambda^2\Re = 0 \tag{1.14}$$

Equation 1.14 can be transformed into the following form

$$\frac{d}{d\eta}\left(\eta\frac{d\Re}{d\eta}\right) + \lambda^2\eta\Re = 0 \tag{1.15}$$

The boundary conditions for Equation 1.15 are as follows

$$\Re'(0) = 0 \tag{1.16}$$

$$\Re(1) = 0 \tag{1.17}$$

The following form of ODEs is commonly referred to as Sturm-Liouville system, where $w = \eta$ is the weighting function. Equation 1.15 is evaluated in Maple 2022 resulting in the following expression

$$\Re(\eta) = AJ_o(\lambda\eta) + BY_o(\lambda\eta) \tag{1.18}$$

Equation 1.18 once differentiated with respect to η yields

$$\Re'(\eta) = -A\lambda J_1(\lambda\eta) - B\lambda Y_1(\lambda\eta) \tag{1.19}$$

From the boundary condition $\Re'(0) = 0$, the following can be obtained

$$\Re'(0) = 0 \text{ as } \eta \to 0, J_1 \to 0, Y_1 \to \infty \tag{1.20}$$

Thu

$$\Re = AJ_o(\lambda\eta) \tag{1.21}$$

Using the second boundary condition, $\Re(1) = 0$, the eigenvalues can be evaluated as

$$AJ_o(\lambda) = 0 \tag{1.22}$$

Since A≠0, the expression becomes

$$J_o(\lambda_n) = 0 \tag{1.23}$$

The roots of the above equation can be expressed more emphatically as

$$\Re_n = A_n J_o(\lambda_n \eta) \tag{1.24}$$

Infinite number of eignevalues and eigenfunctions are thus possible, where some are expressed as follows

$$\begin{aligned} \lambda_1 &= 2.40483 \qquad \Re_1 = J_o(\lambda_1 \eta) \\ \lambda_2 &= 5.52008 \qquad \Re_2 = J_o(\lambda_2 \eta) \\ \lambda_3 &= 8.65372 \qquad \Re_3 = J_o(\lambda_3 \eta) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_n &= \infty \qquad\qquad \Re_n = J_o(\lambda_n \eta) \end{aligned}$$

$$\theta_n = A_n J_o(\lambda_n \eta) e^{-\lambda_n^2 z^+} \qquad \forall \qquad n = 1, 2, 3, \ldots \tag{1.25}$$

For any linear combination of solution is also a solution, hence power series can be devised for constructing a solution

$$\theta(z^+, \eta) = \sum_{n=1}^{\infty} A_n J_o(\lambda_n \eta) e^{-\lambda_n^2 z^+} \tag{1.26}$$

From the boundary condition $\theta(0, \eta) = 1$, the unknown constant A_n can be obtained as

$$1 = \sum_{n=1}^{\infty} A_n J_o(\lambda_n \eta) \tag{1.27}$$

Equation 1.27 is a Fourier Bessel series. By multiplying it with $\eta J_o(\lambda_n \eta)$ and integrating from 0 to 1, the expression can be deduced as

$$\int_0^1 \eta J_o(\lambda_n \eta) d\eta = A_n \int_o^1 \eta J_o^2(\lambda_n \eta) d\eta \tag{1.28}$$

which ultimately leads to

$$A_n = \frac{\int_0^1 \eta J_o(\lambda_n \eta) d\eta}{\int_o^1 \eta J_o^2(\lambda_n \eta) d\eta} \tag{1.29}$$

$J_o(0, \lambda_n) = 0$ for all values of λ, and thus

$$A_n = \frac{2}{\lambda_n} \frac{1}{J_1(\lambda_n)} \tag{1.30}$$

Hence, the temperature distribution in the circular pipe is given by

$$\theta(z^+, \eta) = 2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \frac{J_o(\lambda_n \eta)}{J_1(\lambda_n)} e^{-\lambda_n^2 z^+} \tag{1.31}$$

The team fluid temperature needs to be determined. Since

$$\theta = \frac{T - T_w}{T_i - T_w}$$
$$\eta = \frac{r}{R}$$
$$v_z = V$$

The mean fluid temperature is expressed as

$$T_m = \frac{\int_A v_z T dA}{\int_A v_z dA} = \frac{\int_0^R VT(2\pi r)dr}{\int_0^R (2\pi r)dr} = \frac{\int_0^1 T\eta d\eta}{\int_0^1 \eta d\eta}$$

$$T_m = \frac{\int_0^1 \left[T_w + (T_i - T_w)\left(2\sum_{n=1}^{\infty} \frac{1}{\lambda_n}\frac{J_o(\lambda_n \eta)}{J_1(\lambda_n)} e^{-\lambda_n^2 z^+}\right)\right]\eta d\eta}{\int_0^1 \eta d\eta}$$

$$T_m = T_w + 4(T_i - T_w)\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} e^{-\lambda_n^2 z^+} \tag{1.32}$$

The dimensionless mean fluid temperature is

$$\theta_m = \frac{T_m - T_w}{T_i - T_w} = 4\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} e^{-\lambda_n^2 z^+} \tag{1.33}$$

$$\theta_m = 4\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} e^{\left[-\lambda_n^2 \frac{4(z/D)}{Pe}\right]} \tag{1.34}$$

where $Pe = Re_D Pr$ is the Peclet number. The heat transfer coefficient about the entrance region of the circular pipe for slug flow subjected to constant wall temperature can further be expressed as

$$h = \frac{k\left(\frac{\partial T}{\partial r}\right)_{r=R}}{T_w - T_m} = \frac{k(T_i - T_w)}{R(T_w - T_m)}\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=1} = \frac{-k}{R\frac{T_m - T_w}{T_i - T_w}}\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=1}$$

$$h = -\frac{k}{R}\frac{1}{\theta_m}\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=1} \tag{1.35}$$

As $\frac{dJ_o(\lambda_n \eta)}{d\eta} = -\lambda_n J_1(\lambda_n \eta)$, the wall temperature gradient $(\frac{\partial \theta}{\partial \eta})_{\eta=1}$ becomes

$$(\frac{\partial \theta}{\partial \eta})_{\eta=1} = \left[\frac{\partial}{\partial \eta}\left(2\sum_{n=1}^{\infty} \frac{1}{\lambda_n}\frac{J_o(\lambda_n \eta)}{J_1(\lambda_n)} e^{-\lambda_n^2 z^+}\right)\right]_{\eta=1} = 2\sum_{n=1}^{\infty} \frac{1}{\lambda_n}\frac{e^{-\lambda_n^2 z^+}}{J_1(\lambda_n)}(-\lambda_n J_1(\lambda_n))$$

$$(\frac{\partial \theta}{\partial \eta})_{\eta=1} = -2\sum_{n=1}^{\infty} e^{(-\lambda_n^2 z^+)} \tag{1.36}$$

The Nusselt number is finally obtained as

$$Nu_D = \frac{h(2R)}{k} = \frac{\sum_{n=1}^{\infty} e^{(-\lambda_n^2 z^+)}}{\sum_{n=1}^{\infty} \frac{e^{(-\lambda_n^2 z^+)}}{\lambda_n^2}} \tag{1.37}$$

or

$$Nu_D = \frac{hD}{k} = \frac{\sum_{n=1}^{\infty} e^{\left[-\frac{4\lambda_n^2 (z/D)}{Pe}\right]}}{\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} e^{\left[-\frac{4\lambda_n^2 (z/D)}{Pe}\right]}} \tag{1.38}$$

The Nusselt number with convergent series can be observed for $z^+ \to \infty$ and thus, the local Nusselt number is

$$\lim_{\zeta \to \infty} Nu_D = \lambda_1^2 = (2.40483)^2 = 5.7831 \tag{1.39}$$

The averaged Nusselt number can also be achieved by considering a control volume on pipe length and applying energy balance The energy balance on the

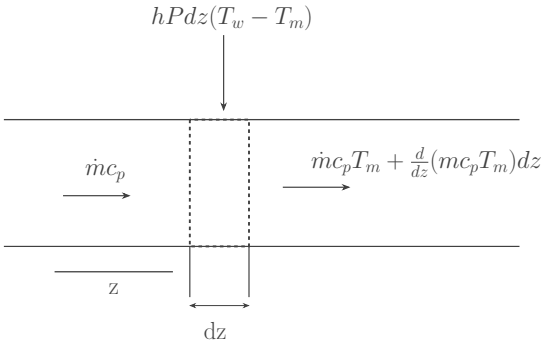


FIGURE 3. Control volume

control volume can be expressed as

$$-\frac{d}{dz}(\dot{m}c_pT_m) + hP(T_w - T_m) = 0 \tag{1.40}$$

$$-\rho V \pi R^2 c_p \frac{dT_m}{dz} + h(2\pi R)(T_w - T_m) = 0 \tag{1.41}$$

$$T_m(0) = T_i \tag{1.42}$$

Equation 1.41 can be expressed in terms of dimensionless variables

$$T_m = T_w + (T_i - T_w)\theta_m$$

$$\frac{dT_m}{dz} = (T_i - T_w)\frac{d\theta_m}{d\xi}\frac{d\xi}{dz} = \frac{\alpha(T_i - T_w)}{VR^2}\frac{d\theta_m}{d\xi}$$

Through considerable manipulation, following expressions are obtained

$$-\frac{d\theta_m}{\theta_m} = Nu_D dz^+ \tag{1.43}$$

$$\tag{1.44}$$

At $z^+ = 0,\ \theta_m = 1$

$$\theta_m = \frac{T_m - T_w}{T_i - T_w} = e^{\left[-\int_0^{z^+} Nu_D dz^+\right]} \tag{1.45}$$

Nu_D is the local Nusselt number on the basis of pipe diameter, the averaged Nusselt number can thsu be defined as

$$\overline{Nu_D} = \frac{\bar{h}D}{k} = \frac{1}{z^+}\int_0^{z^+} Nu_D dz^+ \tag{1.46}$$

Taking natural logarithm of Equation 1.45, using $ln(\theta_m) = -\int_0^{z^+} Nu_D dz^+$, the averaged Nusselt number can be obtained as

$$\overline{Nu_D} = -\frac{ln(\theta_m)}{z^+} \tag{1.47}$$

The local Nusselt number Nu_D, mean Nusselt number $\overline{Nu_D}$, and the mean fluid temperature θ_m can be evaluated in terms of Graetz number.
The concept of logarithmic mean temperature difference can be introduced effectively. Consider the energy balance once again

$$-\dot{m}c_p\frac{dT_m}{dz} + hP(T_w - T_m) = 0 \tag{1.48}$$

The definition for temperature difference as $\Delta T = T_w - T_m$ be introduced, corresponding to which the temperature at the inlet of tube as $\Delta T_i = T_w - T_{mi}$ and outlet of tube $\Delta T_o = T_w - T_{mo}$ be defined. Thus, later can be put in the following form

$$-\frac{dT_m}{dz} = \frac{d\Delta T}{dz} = \frac{hP\Delta T}{\dot{m}c_p} \tag{1.49}$$

By variable separable technique, and performing integration from the tube inlet to the outlet, the expression results in

$$\int_{\Delta T_i}^{\Delta T_o} \frac{d\Delta T}{\Delta T} = -\frac{P}{\dot{m}c_p}\int_0^L h dz \tag{1.50}$$

or

$$ln\left(\frac{\Delta T_o}{\Delta T_i}\right) = -\frac{PL}{\dot{m}c_p}\left(\frac{1}{L}\int_0^L h dz\right) = -\frac{PL\bar{h}}{\dot{m}c_p} \tag{1.51}$$

Considering the definition of local heat transfer coefficient $\bar{h}$, Equation 1.51 becomes

$$\left(\frac{\Delta T_o}{\Delta T_i}\right) = e^{\left[-\frac{PL\bar{h}}{\dot{m}c_p}\right]} \tag{1.52}$$

where the average heat transfer coefficient $\bar{h}$ is

$$\bar{h} = \frac{1}{L}\int_0^L h dz \tag{1.53}$$

The total energy transported by the fluid is given as

$$q = \dot{m}c_p(T_{mo} - T_{mi}) \tag{1.54}$$

Adding and subtracting T_w on both sides of Equation 1.54, the following is obtained

$$q = \dot{m}c_p[(T_w - T_{mi}) - (T_w - T_{mo})] = \dot{m}c_p(\Delta T_i - \Delta T_o) \tag{1.55}$$

Solving Equation 1.52 for $\dot{m}c_p$

$$\dot{m}c_p = \frac{PL\bar{h}}{ln\left(\frac{\Delta T_o}{\Delta T_i}\right)} \tag{1.56}$$

Substituting in Equation 1.55

$$q = PL\bar{h}\frac{-(\Delta T_i - \Delta T_o)}{\left(\frac{\Delta T_o}{\Delta T_i}\right)} \tag{1.57}$$

For averaged heat transfer $\overline{q_w}$

$$\overline{q_w} = \bar{h}A_s\Delta T_{LMTD} \tag{1.58}$$

where ΔT_{LMTD} is the logarithmic mean fluid temperature, $A_s = \pi DL$ is the pipe surface area, L is the pipe length. The logarithmic mean fluid temperature ΔT_{LMTD} is

$$\Delta T_{LMTD} = \frac{(T_w - T_{mi}) - (T_w - T_{mo})}{ln\left[\frac{T_w - T_i}{T_w - T_m}\right]} \tag{1.59}$$

Conclusion

The analytical model presented in the following study successfully characterizes the heat transfer behavior of slug flow in the entrance region of a circular tube under constant wall temperature. Through the simplification of hydrodynamic assumptions and using Bessel-function based solutions, the temperature distribution, mean temperature and Nusselt number correlations were derived in closed form. The results confirm that for low Prandtl number fluids, thermal diffusivity dominates, resulting in rapid temperature field development in comparison with velocity profiles. The local Nusselt number asymptotically approaches $Nu_d = 5.7831$, aligning with the theoretical predictions for fully developed laminar flow. The formulation provides a robust foundation for extending the analysis to cases involving variable wall flux, transient heating or two-phase interactions, making it a relevant document to the advanced thermal management and multiphase system design applications

References

1. , Onur N. (2023). Index. In Introduction to Convective Heat Transfer, N. Onur (Ed.). https://doi.org/10.1002/9781119766797.index

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