AN ANALYTICAL APPROACH TO HEAT TRANSFER FOR SLUG FLOW IN CIRCULAR TUBE SUBJECTED TO CONSTANT HEAT FLUX

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Abstract. This study presents an analytical approach to evaluate heat transfer characteristics in the entrance region of a circular tube subjected to constant wall heat flux under slug flow conditions. The analysis considers a low-Prandtl-number fluid in which the thermal boundary layer develops more rapidly than the hydrodynamic boundary layer, allowing the axial velocity to be treated as constant. Using a two-dimensional, steady, incompressible, laminar flow model, the governing energy equation is solved with the assumption of negligible viscous dissipation and axial conduction. The problem is formulated in dimensionless form, and the temperature field is derived by separation of variables with Bessel functions of the first kind. The resulting expressions for local and average Nusselt numbers are obtained as explicit series solutions in terms of the Graetz number. The analytical solution shows that, at the thermal entrance, the local Nusselt number approaches 8, gradually decreasing along the tube length due to the developing temperature field. The derived relations offer insight into the heat transfer mechanisms in slug flow regimes typical of low-Prandtl-number fluids, and provide a foundational benchmark for validating numerical simulations in microchannel and multiphase heat transport systems.

Keywords. Nusselt number, Prandtl number, thermal diffusivity, Bessel function, Graetz number, viscous dissipation,

1. Circular pipe: Slug Flow Heat Transfer in the Entrance Region

Slug flow is defined as fluid flow in the pipe regime where small air bubble transform to large air bubbles, filling the entire pipe cross-section, with liquid slugs containing smaller entrained gas bubbles present between them. Slug flows are typically observed at low values of Prandtl number.

Considering the short unheated entrance length of tube, with small value of Prandtl number (Pr$<$ 1), where $\delta/\Delta \to 1$, implies that the momentum boundary layer can be ignored from analysis in comparison with the thermal boundary layer. The rate of development of temperature profile is faster than that of velocity profile. The coordinate system is located at the pipe inlet, and the heat transfer associated with slug flow initiates from pipe wall, maintained with uniform heat flux (UHF) as $q_w^{"}$. It can be implies that the constant area duct is heated across the entire length, with thermal boundary later propagating from the entrance of duct and develops progressively. With this, the axial velocity can

be assumed to be constant and it can be approximated as

$$v_z = V$$

where V is the average fluid velocity.

One may consider the two-dimensional,incompressible, steady laminar flow of

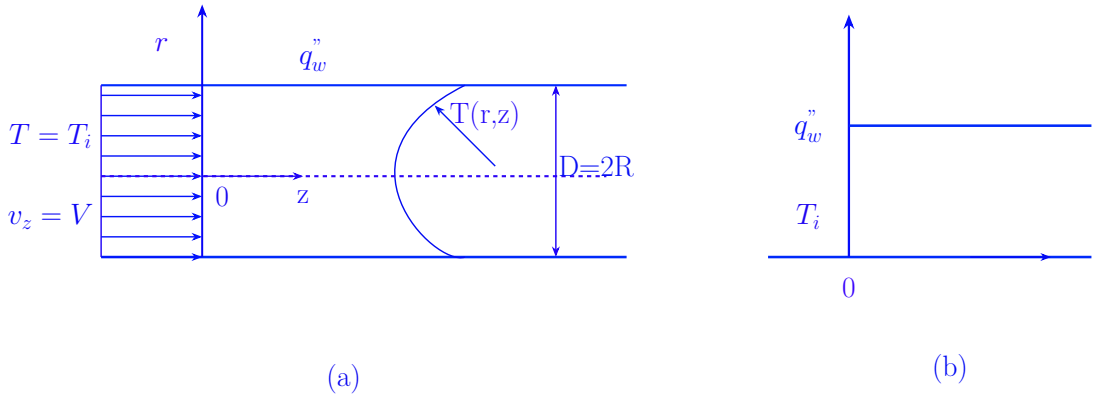


FIGURE 1. (a) Slug flow in circular tube (b) Constant wall temperature boundary condition

a low-Prandtl number fluid flowing in the entrance region of the pipe as seen in 1 The fluid properties are assumed to be constant, with negligible viscous dissipation and no internal energy generation. The pipe radius is denoted as R, where the fluid enters the pipe with an initial temperature of T_i and uniform velocity V. The flow is for low-Prandtl number, where $\delta/\Delta \to 1$, the rate of development of temperature profile is faster than that of velocity profile, implying greater effect of diffusivity. With this, the axial velocity can be assumed to be constant and it can be approximated as $v_z = V$. The coordinate system is located at the duct inlet where the heating begins progressively. Heat flow gradually transforms into slug flow from the thermal pipe wall, which is maintained at constant temperature T_w. The constant area is heated across its entire length, hence it can be construed that the thermal boundary layer development begins at the entrance of the duct and develops along its entire length. Axial conduction and viscous dissipation are ignored from the analysis, with no energy generation. The temperature distribution and the empirical relation for the Nusselt number is to be determined.

The energy equation at the entrance region is to be calculated as

$$V\frac{\partial T}{\partial z} = \alpha\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] \tag{1.1}$$

The corresponding boundary conditions are

$$T(0, r) = T_i \tag{1.2}$$

$$\frac{\partial T(z, 0)}{\partial r} = 0 \tag{1.3}$$

$$\frac{\partial T(z, R)}{\partial r} = \frac{q_w^{"}}{k} \tag{1.4}$$

The dimensionless variables for broader analysis can be defined here as

$$\theta = \frac{T - T_w}{T_i - T_w} \tag{1.5}$$

$$\eta = \frac{r}{R} \tag{1.6}$$

$$z^+ = \frac{z\alpha}{R^2 V} = \frac{2(z/R)}{RePr} = \frac{4(z/D)}{Pe} = \frac{4}{Gz} \tag{1.7}$$

$$\tag{1.8}$$

T_o is an unknown parameter and hence shall be determined as follows

$$T = T_o\theta + T_1 \tag{1.9}$$

$$\frac{\partial T}{\partial r} = T_o\frac{\partial \theta}{\partial r} = T_o\frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{T_o}{R}\frac{\partial \theta}{\partial \eta} \tag{1.10}$$

Through the wall boundary condition at R

$$\frac{T_o}{R}\frac{\partial \theta}{\partial \eta} = \frac{q_w^{"}}{k} \implies \frac{T_o k}{R q_w^{"}}\frac{\partial \theta}{\partial \eta} = 1 \tag{1.11}$$

$$\frac{T_o k}{R q_w^{"}} = 1 \implies T_o = \frac{R q_w^{"}}{k} \tag{1.12}$$

Dimensionless temperature θ becomes

$$\theta(z^+, \eta) = \frac{T(z^+, r) - T_i}{\frac{R q_w^{"}}{k}} \tag{1.13}$$

Using the dimensionless variables, the PDE and corresponding boundary conditions reduce to the following format

$$\frac{\partial \theta}{\partial z^+} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\partial \eta}\right) \tag{1.14}$$

or

$$\frac{\partial \theta}{\partial z^+} = \frac{1}{\eta}\frac{\partial \theta}{\partial \eta} + \frac{\partial^2 \theta}{\partial \eta^2} \tag{1.15}$$

$$z^+ = 0 \qquad \theta = 0 \tag{1.16}$$

$$z^+ > 0 \qquad \eta = 1 \qquad \frac{\partial \theta}{\partial \eta} = 1 \tag{1.17}$$

$$z^+ > 0 \qquad \eta = 0 \qquad \frac{\partial \theta}{\partial \eta} = 0 \text{ or } \theta \text{ is finite} \tag{1.18}$$

$$\tag{1.19}$$

The PDE is not homogeneous and the solution is only possible through technique adopted by [2]

$$\theta(z^+, \eta) = \Psi(z^+, \eta) + \phi(\eta) + \varphi(z^+) \tag{1.20}$$

where $\Psi(z^+, \eta)$ is decaying initial transient value, $\phi(\eta)$ is the radial temperature variation to allow wall heat flux to penetrate into the fluid and $\varphi(z^+)$ is the

increase in axial temperature due to accumulation of heat flux. The formulation ultimately results in one PDE and two ODEs, where the PDE is

$$\frac{\partial \Psi}{\partial z^+} = \frac{\partial^2 \Psi}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \Psi}{\partial \eta} \tag{1.21}$$

$$\Psi(0,\eta) = -\phi(\eta) - \varphi(0) \tag{1.22}$$

$$z^+ > 0 \qquad \eta = 1 \qquad \frac{\partial \Psi}{\partial \eta} = 1 \tag{1.23}$$

$$z^+ > 0 \qquad \eta = 0 \qquad \frac{\partial \Psi}{\partial \eta} = 0 \text{ or } \Psi \text{ is finite} \tag{1.24}$$

$$\tag{1.25}$$

The ODEs are

$$\frac{\partial \varphi}{\partial z^+} = \frac{\partial^2 \phi}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \phi}{\partial \eta} \tag{1.26}$$

$$\eta = 0 \qquad \phi = \text{finite}$$

$$\eta = 1 \qquad \frac{d\phi}{d\eta} = 1$$

Variables $\phi(\eta)$ and $\varphi(z^+)$ can independently vary, the equation is consistent and meaningful provided it is equal to a constant C_4

$$\varphi(z^+) = C_4 z^+ + C_5 \tag{1.27}$$

$$phi(\eta) = \frac{1}{4}C_4\eta^2 + C_6 ln(\eta) + C_7 \tag{1.28}$$

Using the boundary conditions, $\theta(z^+, 0) = \text{finite}$ implies $C_6 = 0$

$$\varphi(z^+) = C_4 z^+ + C_5 \tag{1.29}$$

$$phi(\eta) = \frac{1}{4}C_4\eta^2 + C_7 \tag{1.30}$$

Using the boundary condition $\left(\frac{\partial \phi}{\partial \eta}\right)_{\eta=1}$ results in $C_4 = 2$, thus the equation becomes

$$\varphi(z^+) = 2z^+ + C_5 \tag{1.31}$$

$$\phi(\eta) = \frac{1}{2}\eta^2 + C_7 \tag{1.32}$$

The solution of $\Psi(z^+, \eta)$ is only dependent on $\phi(\eta)$ and $\varphi(z^+)$, the constant terms C_5 and C_7 may be set to zero value. Hence,

$$\varphi(z^+) = 2z^+ \tag{1.33}$$

$$\phi(\eta) = \frac{1}{2}\eta^2 \tag{1.34}$$

On the other hand, the product solution of following form can be assumed as

$$\Psi = \Re(\eta) Z(z^+) \tag{1.35}$$

Substituting 1.35 in the 1.21 and dividing by $\Re(\eta)Z(z^+)$ and separating variables. following may be obtained

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\Re}{d\eta}\right) = \frac{1}{Z}\frac{dZ}{dz^+} = -\lambda^2 \tag{1.36}$$

or

$$\eta^2\frac{d^2\Re}{d\eta^2} + \eta\frac{d\Re}{d\eta} + \lambda^2\eta^2\Re = 0 \tag{1.37}$$

$$\frac{dZ}{dz^+} + \lambda^2 Z = 0 \tag{1.38}$$

The solutions of Equations 1.37 and 1.38 were evaluated in Maple with Bessel Functions of zeroth kind

$$\Re = C_1 J_0(\lambda\eta) + C_2 Y_o(\lambda\eta) \tag{1.39}$$

$$Z = C_3 e^{-\lambda^2 z^+} \tag{1.40}$$

Using the boundary conditions, $\theta(z^+, 0) =$ finite gives $\Re(0) =$ finite, which ultimately leads to $C_2 = 0$ as $Y_0 \to -\infty$ for $\eta \to 0$. Ultimately, the solution becomes

$$\Re = C_1 J_o(\lambda\eta) \tag{1.41}$$

The boundary condition at $\eta = 1$, $\Re = 0$ requirement results in an infinite number of roots of

$$\frac{d\Re}{d\eta} = C_1\frac{d}{d\eta}[J_o(\lambda\eta)]_{\eta=1} = 0 \tag{1.42}$$

or

$$J_1(\lambda_n) = 0 \tag{1.43}$$

Solving Equation 1.43 for determining a few values of eignevalues as

$$\lambda_0 = 0$$
$$\lambda_1 = 3.8317$$
$$\lambda_2 = 7.0156$$
$$\lambda_3 = 10.1735$$
$$\lambda_1 = 13.3237$$
$$\lambda_1 = 16.4706$$

In terms of product solution

$$\Psi = \sum_{n=0}^{\infty} a_n \exp(-\lambda_n^2 z^+) J_o(\lambda_n\eta) \tag{1.44}$$

where $a_n = C_1C_3$. The initial condition $\Psi(0, \eta) = -\phi(\eta) - \varphi(\eta)$ be applied

$$-\frac{1}{2}\eta^2 = a_o + \sum_{n=0}^{\infty} a_n J_o(\lambda_n\eta) \tag{1.45}$$

The integrals can be mathematically evaluated as

$$a_o = 2\int_0^1 \eta\left(-\frac{\eta^2}{2}\right)d\eta = -\frac{1}{4} \tag{1.46}$$

$$a_n = \frac{\int_o^1 \eta\left(-\frac{\eta^2}{2}\right)J_o(\lambda_n\eta)d\eta}{\int_0^1 \eta J_o^2(\lambda_n\eta)d\eta} \tag{1.47}$$

Since $J_1(\lambda_n) = 0$, the above expression becomes

$$a_n = \frac{\int_o^1 \eta\left(-\frac{\eta^2}{2}\right)J_o(\lambda_n\eta)d\eta}{\int_0^1 \eta J_o^2(\lambda_n\eta)d\eta} = -\frac{2}{\lambda_n^2 J_o(\lambda_n)}\forall n \neq 0 \tag{1.48}$$

The final result for the dimensionless temperature is

$$\theta = 2z^+ + \frac{1}{2}\eta^2 - \frac{1}{4} - 2\sum_{n=1}^{\infty}\exp(-\lambda_n^2 z^+)\frac{J_o(\lambda_n\eta)}{J_o(\lambda_n)\lambda_n^2} \tag{1.49}$$

The mixing cub fluid temperature is

$$T_m = \frac{\int_0^1 V\left[T_i + \frac{q_w^{"}R}{k}\left(2z^+ + \frac{1}{2}\eta^2 - \frac{1}{4} - 2\sum_{n=1}^{\infty}\exp(-\lambda_n^2 z^+)\frac{J_o(\lambda_n\eta)}{J_o(\lambda_n)\lambda_n^2}\right)\right](2\pi\eta d\eta)}{\int_0^1 V(2\pi\eta)d\eta} \tag{1.50}$$

Through integration and algebraic manipulation in Maple, the mean fluid temperature is obtained

$$T_m = 2\left(\frac{q_w^{"}R}{k}\right)z^+ + T_i \tag{1.51}$$

in dimensionless form

$$\theta_m = \frac{T_m - T_i}{\left(\frac{q_w^{"}R}{k}\right)} = 2z^+ \tag{1.52}$$

The wall temperature is obtained by setting $\eta = 1$ as

$$\theta_w = \theta(z^+, 1) = 2z^+ + \frac{1}{4} - 2\sum_{n=1}^{\infty}\frac{\exp(-\lambda_n^2 z^+)}{\lambda_n^2} \tag{1.53}$$

The difference between θ_m and θ_w is

$$\theta_w - \theta_m = \frac{1}{4} - 2\sum_{n=1}^{\infty}\exp(-\lambda_n^2 z^+) \tag{1.54}$$

The local heat transfer coefficient has to be defined as

$$h(T_w - T_m) = q_w^{"} \tag{1.55}$$

or

$$h(T_w - T_m) = k\left(\frac{\partial T}{\partial r}\right)_{r=R} \tag{1.56}$$

In dimensionless form

$$\frac{hR}{k}(\theta_w - \theta_m) = \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \tag{1.57}$$

$$\frac{h(2R)}{k}\frac{(\theta_w-\theta_m)}{2}=\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \tag{1.58}$$

Evaluating $\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}$, it can be interpreted that

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}=1 \tag{1.59}$$

The local Nusselt number Nu_D is

$$Nu_D=\frac{hD}{k}=\frac{2}{(T_w-T_m)} \tag{1.60}$$

Using Equation 1.54 and Equation 1.60, the local Nusselt number Nu_D can be obatined

$$Nu_D=\frac{8}{1-8\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\exp(-\lambda_n^2 z^+)} \tag{1.61}$$

For large values of z^+

$$\lim_{z^+\to 0} Nu_D=8 \tag{1.62}$$

The average Nusselt number $\overline{Nu_D}$ is typically defined on the basis of averaged difference between wall and mixing cup temperatures,

$$\overline{T_w}-\overline{T_m}=\frac{1}{z^+}\int_0^{z^+}(T_w-T_m)d\xi \tag{1.63}$$

Where ξ is the dummy variable for integration. The averaged heat transfer coefficient $\bar{h}$ is

$$\bar{h}=\frac{q_w^{"}}{\overline{T_w}-\overline{T_m}} \tag{1.64}$$

The averaged Nusselt number $\overline{Nu_D}$ becomes

$$\overline{Nu_D}=\frac{\bar{h}D}{k}=\frac{D}{k}\frac{q_w^{"}}{\overline{T_w}-\overline{T_m}}=\frac{2}{\overline{\theta_w}-\overline{\theta_m}} \tag{1.65}$$

where $\overline{\theta_w}-\overline{\theta_m}$ is

$$\overline{\theta_w}-\overline{\theta_m}=\frac{1}{z^+}\int_0^{z^+}(\overline{\theta_w}-\overline{\theta_m})d\xi \tag{1.66}$$

Substituting Equation 1.54 in 1.66 to obtain

$$\overline{\theta_w}-\overline{\theta_m}=\frac{1}{z^+}\int_0^{z^+}\left[\frac{1}{4}-2\sum_{n=1}^{\infty}\frac{e^{-\lambda_n^2\xi}}{\lambda_n^2}\right]d\xi \tag{1.67}$$

$$\overline{\theta_w}-\overline{\theta_m}=\frac{1}{z^+}\left[\frac{z^+}{4}-2\sum_{n=1}^{\infty}\frac{1-e^{-\lambda_n^2 z^+}}{\lambda_n^4 z^+}\right]_0^{z^+} \tag{1.68}$$

$$\overline{\theta_w}-\overline{\theta_m}=\frac{1}{4}-2\sum_{n=1}^{\infty}\left[\frac{1-e^{-\lambda_n^2 z^+}}{\lambda_n^4 z^+}\right] \tag{1.69}$$

The average Nusselt number becomes

$$\overline{Nu_D} = \frac{8}{1 - 8\sum_{n=1}^{\infty}\left[\frac{1-e^{-\lambda_n^2 z^+}}{\lambda_n^4 z^+}\right]} \tag{1.70}$$

Conclusion

An analytical solution for heat transfer in the entrance region of a circular tube subjected to uniform wall heat flux has been successfully derived for slug flow of a low-Prandtl-number fluid. The formulation, based on the steady-state energy equation and solved using Bessel functions, yielded explicit expressions for both local and average Nusselt numbers as a function of the dimensionless axial coordinate. The analysis confirms that for the initial section of the tube, the local Nusselt number tends to a limiting value of 8, consistent with theoretical expectations for fully developed slug flow. The temperature distribution and wall-to-fluid temperature difference evolve progressively along the tube, highlighting the dominance of thermal diffusivity in low-Prandtl-number conditions. These results provide a rigorous analytical framework for assessing heat transfer in slug flow systems and can serve as a reference for experimental and computational validation of thermally developing flows in compact heat exchangers and multiphase transport applications.

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