One-Dimensional Solutions in Convective Heat Transfer: Heat Transfer in Poisuelle Flow

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Abstract

This study investigates the coupled convective heat transfer and viscous dissipation in Poiseuille flow configuration between two infinite parallel plates. The upper plate and lower plate remain stationary and is maintained at a much lower temperature than the upper plate. A one-dimensional analytical solution is derived for velocity and temperature distributions under laminar, steady, and incompressible flow conditions with constant fluid properties. The governing equations, including the momentum and energy equations, are simplified using boundary-layer assumptions and solved symbolically in MATLAB. Dimensionless expressions for temperature profile and heat flux are obtained as functions of the Brinkman and Eckert numbers. The results provide insights into the influence of viscous dissipation and Brinkman number on heat transport phenomena, with implications for design and analysis of microfluidic devices and thermal systems where viscous heating is non-negligible.

Keywords: Poiseulle flow; viscous dissipation; Brinkman number; heat transfer; Nusselt number; MATLAB modeling

Poiseulle flow is the steady flow of hydrodynamically fully developed laminar flow of an incompressible fluid through a cross-sectional pipe with rotational symmetry Kano (2013). They are generally pressure driven flows along the length of a channel with fixed cross-section Rapp (2017). The flows are assumed to be parallel and usually are restricted to incompressible fluids. Poiseulle flows are extensively affiliated with many microfluidic applications where a fluid is probed along a channel structure with a given cross-section shape. The fluid will interact with the boundary of the cross-section ∂C where the fluid's velocity is usually assumed to be 0 (no-slip condition). Streamlines for Poiseulle flows are typically parallel to channnel walls, with the flow associated with French scientist J. L. M. Poiseulle White and Majdalani (2021). Consider 1, consisting of two infinite parallel

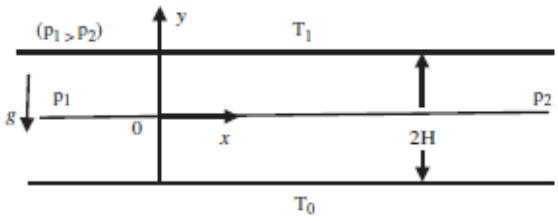


Figure 1: Heat Transfer in Poiseulle Flow Onur (2023)

impermeable plates. An incompressible, constant property fluids fill the space between the plates. The upper and lower plates are kept stationary The temperature of lower plate is lesser than that of upper plate ($T_1 > T_0$). The fluid flow is assosiated with development of pressure gradient in the axial flow direction, thus $\frac{dp}{dx} =$ constant pressure gradients exist in the flow. Considering the viscous dissipation into account, the velocity distribution, temperature distribution, heat flux at both surfaces and location of maximum temperature can be evaluated for lower surface while neglecting the influence of gravity or pressure variation in fluid.

As stated above, the plates are of infinite length and there occurs no variation in x- or z- direction and there is no motion in z-direction as well, thus $w = 0$. The viscous dissipation in flow leads to generation of energy and the fluid flow tends to fully develop since all fluid particles move parallel to plates. The velocity distribution, once determined ultimately leads to determination of temperature distribution, heat flux and Nusselt number. For effective analysis of phenomenon, following assumptions may be taken into account:

- Flow remains steady throughout
- The flow rests in laminar regime
- The fluidic properties such as density ρ, heat capacity c_p, conductivity k and viscosity (dynamic, μ) remain constant throughout.
- Since plates are infinite, so the edge effects are ignored $\frac{\partial}{\partial z} = 0$
- Since plates are infinite, so the end effects are ignored $\frac{\partial}{\partial x} = 0$, however the pressure gradient is never zero, i.e $\frac{\partial p}{\partial x} \neq 0$
- The fluid around plates is Newtonian
- Pressure gradient along the horizontal direction and continuum should be satisfied
- There is no motion in z-direction, $w = 0$

The continuity equation can be expressed as

$$\frac{D\rho}{Dt} + \rho(\vec{\nabla}.\vec{V}) = 0 \tag{1}$$

or

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \tag{2}$$

For an incompressible fluid

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial y} = \frac{\partial \rho}{\partial z} = 0$$

The final continuity equation becomes

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

Since the two parallel plates are of infinite length, the edge and end effects are not taken into consideration, and since there is no motion in z-direction as $w = 0$, so

$$\frac{\partial w}{\partial z} = 0 \qquad \text{no edge effects}$$
$$\frac{\partial u}{\partial x} = 0 \qquad \text{No end effects and hydrodynamically fully developed fluid flow}$$

From the above simplifications, the continuity equation reduces to following format:

$$\frac{\partial v}{\partial y} = 0$$

On the basis of such simplifications, the continuity equation reduces to

$$v(x, y) = f(x)$$

By the no-slip boundary condition, since it is applied at the walls

$$y = 0 \qquad v = 0 \qquad \text{Impermeable wall}$$

which implies $f(x) = 0$, so the y-component of velocity becomes

$$v(x, y) = 0$$

which further implies that the streamlines are parallel in flow field, and the y-component of velocity is zero elsewhere. Considering the y-component in the momentum equation

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho f_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

The pressure term reduces to

$$\frac{dp}{dy} = -\rho g$$

By integration

$$p = p_o - \rho g y$$

Considering the x-component of momentum equation

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$
\begin{aligned}
f_x &= 0 \\
\frac{\partial u}{\partial t} &= 0 \qquad \text{Steady Flow} \\
\frac{\partial u}{\partial x} &= 0 \qquad \text{Hydrodynamically fully developed flow and no end effects} \\
\frac{\partial u}{\partial y} &= 0 \qquad \text{v=0} \\
\frac{\partial u}{\partial z} &= 0 \qquad \text{No edge effects and considering no motion in z-direction, thus w=0} \\
\frac{\partial^2 u}{\partial x^2} &= 0 \qquad \text{since } \frac{\partial u}{\partial x} = 0 \\
\frac{\partial^2 u}{\partial z^2} &= 0 \qquad \text{since } \frac{\partial u}{\partial z} = 0 \\
\frac{\partial p}{\partial x} &= \text{constant} \qquad \text{Pressure gradient is constant throughout}
\end{aligned}
$$

The horizontal component of momentum equation reduces to

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dx}$$

The corresponding boundary conditions are:

$$
\begin{aligned}
u(-H) &= 0 \qquad \text{No slip effect} \\
u(H) &= U \qquad \text{No slip effect}
\end{aligned}
$$

Once plugged in MATLAB solver using the following code, the resultant expression becomes:

```
clc;
clear all;
syms u(y) y h U k T(y) mu F T0 T1

de=diff(u,y,2)==(1/mu)*(dp/dx);

BC(1)=u(-H)==0;
BC(2)=u(H)==0;

BC=[BC(1),BC(2)]

u=dsolve(de,BC)
u=expans(u)
```

Listing 1: MATLAB code for evaluating the velocity profile

$$\frac{\mathrm{dp}\, y^2}{2\,\mathrm{dx}\,\mu} - \frac{H^2\,\mathrm{dp}}{2\,\mathrm{dx}\,\mu}$$

which can further be simplified as

$$\frac{1}{2}\frac{dpy^2}{\mu dx} - \frac{1}{2}\frac{dpHy}{\mu dx}$$

Further simplification leads tp

$$u = \left(\frac{-dp}{dx}\right)\left(\frac{H^2}{2\mu}\right)\left[1 - \left(\frac{y}{H}\right)^2\right]$$

The average fluid velocity is given as

$$V = \frac{1}{2H}\int_{-H}^{H} u dy$$

By integration

$$V = \left(\frac{H^2}{3\mu}\right)\left(\frac{-dp}{dx}\right)$$

About the centerline, i.e, y=0, maximum velocity can be observed

$$V_c = \left(\frac{H^2}{2\mu}\right)\left(\frac{-dp}{dx}\right)$$

The center velocity V_c can be expressed in terms of average fluid velocity V such

$$V_c = \frac{3}{2}V$$

Hence, the velocity can be written as

$$u = V_c\left[1-\left(\frac{y}{H}\right)^2\right] = \frac{3V}{2}\left[1-\left(\frac{y}{H}\right)^2\right]$$

From the velocity distribution once known, the energy equation coupled with viscous dissipation can be taken into consideration:

$$\rho c_p\left[\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right] = k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]+q'''+\mu\Phi$$

where

$$\Phi = \left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2$$

$\frac{\partial T}{\partial t}=0$	Steady Flow
$\frac{\partial T}{\partial x}=0$	No end effects
$\frac{\partial u}{\partial x}=0$	No end effects, fully developed hydrodynamic laminar flow
$\frac{\partial T}{\partial z}=0$	No edge effect and w=0 since there is no motion in z-direction
$v\frac{\partial T}{\partial y}=0$	Since v=0 due to continuity equation
$\frac{\partial^2 T}{\partial x^2}=0$	Negligible axial conduction
$\frac{\partial^2 T}{\partial z^2}=0$	No edge effects. Since there are no edge effects so $\frac{\partial T}{\partial z}=0$and since$\frac{\partial T}{\partial z}=0$, $\frac{\partial^2 T}{\partial z^2}=0$

From all the following aspects, the energy equation reduces to the following expression

$$k\frac{d^2T}{dy^2}+\mu\Phi = 0$$

Considering the assumptions made, the viscous dissipation term reduces to

$$\Phi = \left(\frac{\partial u}{\partial y}\right)^2$$

The corresponding boundary conditions are

$$T(-H) = T_o$$
$$T(H) = T_1$$

```
  clc
clear all
syms u(y) y dp dx mu k H l Vc T(y) T0 T1

u=Vc*(1-(y/H)^2)
F=diff(u,y)^2

de=diff(T,y,2)+(mu/k)*F==0

BC1=T(-H)==T0
BC2=T(H)==T1
BC=[BC1,BC2]

sol = dsolve(de, BC);
var=vpa(sol);
sol = expand(sol);
```

Listing 2: MATLAB code for evaluating temperature profile

On running the above MATLAB following equation is yielded

$$\frac{2\,\mu\,\mathrm{Vc}^2 + 3\,T_0\,k + 3\,T_1\,k}{6\,k} - \frac{y\,(T_0 - T_1)}{2\,H} - \frac{\mathrm{Vc}^2\,\mu\,y^4}{3\,H^4\,k}$$

On further simplification

$$\frac{T_0}{2} + \frac{T_1}{2} + \frac{\mathrm{Vc}^2\,\mu}{3\,k} - \frac{T_0\,y}{2\,H} + \frac{T_1\,y}{2\,H} - \frac{\mathrm{Vc}^2\,\mu\,y^4}{3\,H^4\,k}$$

The final solution can be expressed as

$$T = T_o + \left(1 + \frac{y}{H}\right)\left[\frac{T_1 - T_o}{2}\right] + \left(\frac{\mu V_c^2}{3k}\right)\left[1 - \left(\frac{y^4}{H^4}\right)\right]$$

Incorporating the dimensionless numbers, Eckert and Prandtl, the expression can be reduced as

$$\frac{T - T_o}{T_1 - T_o} = \frac{1}{2}\left[1 + \frac{y}{H}\right] + \frac{EcPr}{3}\left[1 - \left(\frac{y}{H}\right)^4\right]$$

In dimensionless form, the expression reduces to

$$\theta = \frac{1}{2}(1 + \eta) + \frac{EcPr}{3}(1 - \eta^4)$$

where $\theta = \frac{T - T_o}{T_1 - T_o}$, $\eta = \frac{y}{H}$, $Pr = \frac{\mu c_p}{k}$ and $Ec = \frac{U^2}{c_p(T_1 - T_o)}$is Eckert number. Eckert number is the measure of viscous dissipation in fluid. The product of Eckert number and Prandtl number is referred as Brinkman]n number, $Br = EcPr$. For determining the location of maximum temperature, $\frac{dT}{dy} = 0$ and solution be initiated for y

$$\frac{dT}{dy} = \frac{1}{2}\left[\frac{T_1 - T_0}{H}\right] - \frac{(T_1 - T_0)EcPr}{3}\frac{4y_{max}^3}{H^4} = 0$$

The second derivative of temperature $\frac{d^2T}{dy^2} < 0$, thus the solution results in

$$y_{max} = \sqrt[3]{\frac{3}{8(Pr)(Ec)}}$$

Substituting the above expression in temperature distribution expression,

$$\frac{T_{max} - T_o}{T_1 - T_o} = \frac{1}{2}\left[1 + \left(\frac{3}{8}\frac{1}{Ec.Pr}\right)^{\frac{1}{3}}\right] + \frac{EcPr}{3}\left[1 - \left(\frac{3}{8}\frac{1}{Ec.Pr}\right)^{\frac{4}{3}}\right]$$

For $T_1 > T_o$, the heat flux can be evaluated at both ends of plates.

$$\frac{dT}{dy} = \left[\frac{1}{2H} - \frac{EcPr}{3}\frac{4y^3}{H^4}\right](T_1 - T_0)$$

For $T_1 > T_o$, the heat flux can be evaluated at both ends of plates. For lower plate, the corresponding expression for heat flux can be written as

$$q_w^{"} = -k\left(\frac{dT}{dy}\right)_{y=-H} = -k\left(\frac{T_1 - T_0}{2H}\right)\left(1 + \frac{8}{3}PrEc\right)$$

For the upper moving plate, the heat flux equation can be written as

$$q_w^{"} = -k\left(\frac{dT}{dy}\right)_{y=H} = -k\left(\frac{T_1 - T_0}{2H}\right)\left(1 - \frac{8}{3}PrEc\right)$$

Considering the heat transfer at lower plate, the heat transfer takes place from fluid to the plate. Where Brinkmann number $Br = EcPr$, the plots have been devised for $EcPr$ Considering 2, the following cases can be established for heat transfer at the upper plate (considering the blue curve):

- For $EcPr < 0.375$, heat transfer takes place from plate to the fluid
- For $EcPr > 0.375$, heat transfer takes place from fluid to the plate
- For $EcPr < 0.375$, no heat transfer takes place

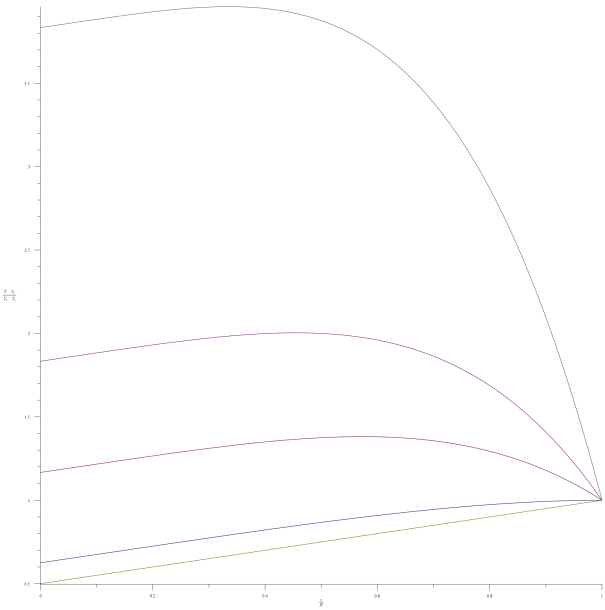


Figure 2: Dimensionless temperature distribution in Poisuelle flow for different Brinkman numbers.

Conclusion

In this study, a comprehensive analytical formulation for heat transfer in Poiseulle flow with viscous dissipation was developed. The flow field and thermal behavior between two infinite parallel plates were examined under a set of simplifying assumptions, allowing for an exact solution to the coupled momentum and energy equations. The results demonstrate that the temperature distribution in the fluid is strongly influenced by the Eckert and Prandtl numbers, with significant deviations from linear profiles at higher viscous dissipation levels. T

The analysis confirms that for Brinkman number (the product of Eckert and Prandtl number $Br = EcPr$) greater than 0.375, heat is transferred from the fluid to the moving plate, whereas for values less than 0.375, the reverse occurs. A special condition arises at $Br = 0.375$ where no net heat transfer takes place at the upper boundary. These findings underscore the importance of accounting for viscous effects in thermal systems, especially in microscale applications and highly viscous flows. The analytical framework presented here provides a reliable benchmark for validating numerical solvers and optimizing thermal-fluid designs.

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