Analysis of Fluid Flow over Flat Plate using Navier Stokes and Continuity Equations

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Abstract

This study presents a numerical analysis of fluid flow over a plate using the Navier-Stokes and Continuity equations, with and without the consideration of heat flux. The governing equations are discretized using the finite difference approximation method to obtain velocity and temperature distributions across the computational domain. The study assumes an incompressible, two-dimensional, steady-state flow with uniform velocity conditions and no pressure gradient effects. For cases including heat flux, the energy equation is coupled with the Navier-Stokes equations to analyze temperature variations and the development of the thermal boundary layer. Contour plots of velocity and temperature distributions illustrate the influence of viscosity and heat transfer on flow behavior. Results show the formation of a boundary layer near the plate, with velocity increasing from zero at the surface to the free-stream value, confirming the classic boundary layer theory. In the presence of heat flux, thermal diffusion leads to significant temperature gradients near the plate, demonstrating the coupling between flow dynamics and heat transfer.

Keywords: Navier Stokes Equation; Continuity Equation; discretization; finite difference approximation; thermal boundary layer; heat flux; thermal diffusion

1 Introduction

Navier-Stokes equations are partial differential equations used to describe the motion of viscous fluids. named after Claude Louis Navier, a French scientist and George Gabriel Stokes, an Irish mathematician, the Navier-Stokes equations evolved progressively over time, from 1822 by Navier and between 1842-1850 by Stokes.
Navier-Stokes equations mathematically express the momentum balance for the conservation of mass, while sometimes accompanied by equations of state such as pressure, temperature and density. They arise from applying Newton's second law of motion to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term—hence describing viscous flow McLean (2012). The difference between them and the closely related Euler equations is that Navier–Stokes equations take viscosity into account while the Euler equations model only inviscid flow. As a result, the Navier–Stokes are an elliptic equation and therefore have better analytic properties, at the expense of having less mathematical structure (e.g. they are never completely integrable). The Navier-Stokes equation in incompressible form can with uniform viscosity in convective form:

$$\frac{dU}{dt} = \frac{\partial u}{\partial t} + (u.\nabla)u = \nu\nabla^2 u - \frac{1}{\rho}\nabla p + \frac{1}{\rho}f \tag{1}$$

Where ν=μ/ρ is known as the kinematic viscosity. By isolation of fluid velocity, the Navier Stokes equation for incompressible flow can be expressed for constant velocity as:

$$(\frac{\partial}{\partial t} + u.\nabla - \nu\nabla^2)u = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}f \tag{2}$$

Assuming constant viscosity and density, in convective form, the Navier Stokes can be represented as:

$$\frac{Du}{Dt} = \nu\nabla^2 u - \nabla\frac{p}{\rho} + \frac{1}{\rho}f \tag{3}$$

BY taking curl of incompressible form of Navier Stokes Equation, the elimination of pressure occurs, with simplification resulting in assuming the 2D Cartesian fluid flow where equations reduce to:

$$\rho(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}) = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}) + \rho g_x \tag{4}$$

$$\rho(\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y}) = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}) + \rho g_y \tag{5}$$

2 Analysis of Fluid Flow over Plate (Without considering Heat Flux)

For the sake of simplification here , fluid flow over a film shall be analyzed by coupling of Navier Stokes equation and Continuity Equation where following conditions have been brought into consideration:

1. Incompressible fluid flow
2. 2D fluid flow
3. The velocity stream remians constant throughout
4. There is no pressure drop neither gravity influences the flow
5. For boundary layers, there is no slip condition for u=0 in solid plate and U=u_o from solid plate where uniform fluid flow is observed for u=u_o.

Here the solution strategy can be devised such that the Continuity equation is expressed as:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{6}$$

and the Navier Stokes can be expressed in 2D by simplification as:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}(\frac{\partial^2 u}{\partial y^2}) \tag{7}$$

By the application of Finite Difference Approximation on both the Navier Stokes Equation and Continuity Equation, expressions can be related: By forward and central 1st order FDA on Navier Stokes results in:

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} \tag{8}$$

$$\frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} \tag{9}$$

$$\frac{\partial^2 u}{\partial^2 y} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} \tag{10}$$

The final form of discretized Navier Stokes Equation can be expressed as:

$$u_{i+1,j}(\frac{u_{i+1,j} - u_{i,j}}{\Delta x}) + v_{i,j}(\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}) = \frac{\mu}{\rho}(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2}) \tag{11}$$

or,

$$u_{i+1,j} = u_{i,j} + \frac{\mu}{\rho}(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2})\frac{\Delta x}{u_{i,j}} - (v_{i,j}\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y})\frac{\Delta x}{u_{i,j}} \tag{12}$$

Similarly for the continuity equation, discretization can be performed such that:

$$\frac{v_{i+1,j} - v_{i,j-1}}{\Delta y} = \frac{-1}{2}(\frac{u_{i+1,j-1} - u_{i,j-1}}{\Delta x} + \frac{u_{i+1,j} - u_{i,j}}{\Delta x}) \tag{13}$$

The final form of discretized Continuity Equation can be expressed as:

$$v_{i+1,j} = v_{i+1,j-1} - \frac{\Delta y}{2}((\frac{u_{i+1,j-1} - u_{i,j-1}}{\Delta x} + \frac{u_{i+1,j} - u_{i,j}}{\Delta x}) \tag{14}$$

```
clc
clear
close all
\%required data and properties of ambient air
u0=7; \%freestream velocity of air
nu=1.48e-5; \%kinemtic viscosity of air (nu=mu/rho)
\%dimesnions of horizontal plate
X=1.0; \%Length of plate
Y=0.02; \%horizontal distance from plate
```

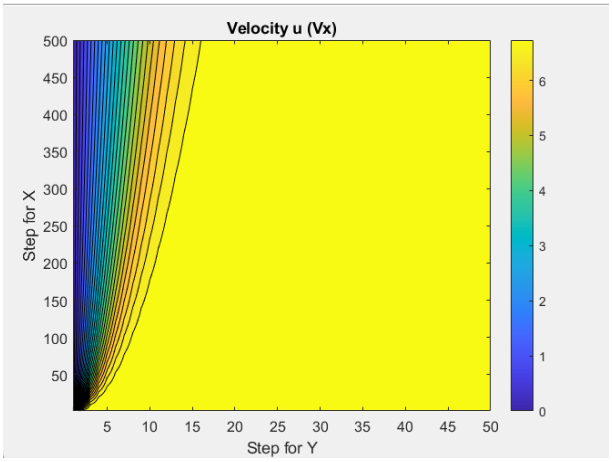
```
    \%Step size and discretization
    imax=500; \%Step size for x
    jmax=50; \%stepsize for y
    dx=X/imax;
    dy=Y/jmax;
    \%Defination of Matrix
    u=ones(imax,jmax); \%For u(Vx)
    x=zeros (1,imax); \%Defination of length of X
    v=zeros(imax,jmax); \%For v(Vy)
    y=zeros(1,jmax); \%Defination of length of Y
    \%Application of Boundary Conditions
    u(:,:)=1;
    u(:,1)=0;
    u(1,:)=u0;
    u(:,jmax)=u0;
    u(1,1)=0;
    v(1,:)=0;
    v(:,1)=0;
    v(1,1)=0;
    \%Discretized Navier Stokes and Continuity Equations
    for o=1:imax
    for j=2:jmax-1
    for i=1:imax-1
    \%Navier Stokes Equation
    u(i+1,j)=u(i,j)+nu*(u(i,j+1)-2*u(i,j)+u(i,j-1)).*dx./u(i,j)./dy.\^2-v(i,j).*(u(i,j+1)
    -u(i,j-1)).*dx/(u(i,j).*2.*dy);
    \%Continuity Equation
    v(i+1,j)=v(i+1,j-1)-(u(i+1,j)-u(i,j)+u(i+1,j-1)-u(i,j-1))*0.5.*dy./dx;
    end
    end
    end
    figure(1)
    contourf(u,25)
    colorbartitle('Velocity u (Vx)')
    xlabel('Step for Y')
    ylabel('Step for X')
    figure(2)
    contourf(v,25)
    colorbartitle('Velocity v (Vy)')
    xlabel ('Step for Y')
    ylabel('Step for X')
    figure(3)
    contour(u,25)
    colorbartitle('Velocity u (Vx)')
    xlabel('Step for Y')
    ylabel('Step for X')
    figure(4)
    contour(v,25)
    colorbartitle('Velocity v (Vy)')
    xlabel ('Step for Y')
    ylabel('Step for X')
```

Listing 1: Analysis of Flow over Plate using Navier Stokes and Continuity Equations (No Heat Flux)

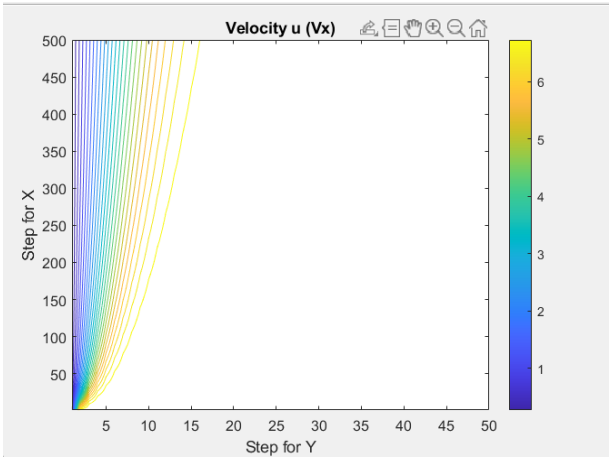
3 Analysis of Fluid Flow over Plate (With considering constant Heat Flux)

For the analysis of fluid flow over plate with considering heat flux, heat transfer equation is coupled with the Navier Stokes Equation and Continuity Equations where following conditions have been brought into consideration:

1. Incompressible fluid flow
2. 2D fluid flow
3. The velocity stream and temperature flux remains constant throughout
4. There is no pressure drop neither gravity influences the flow
5. For boundary layers, there is no slip condition for u=0 in solid plate and U=u_o from solid plate where uniform fluid flow is observed for u=u_o.
6. The wall temperature is T=T_W while being infinitely away from the wall
7. The velocity is uniform hence u=u_∞ and the temperature is uniform such that T=T_∞.

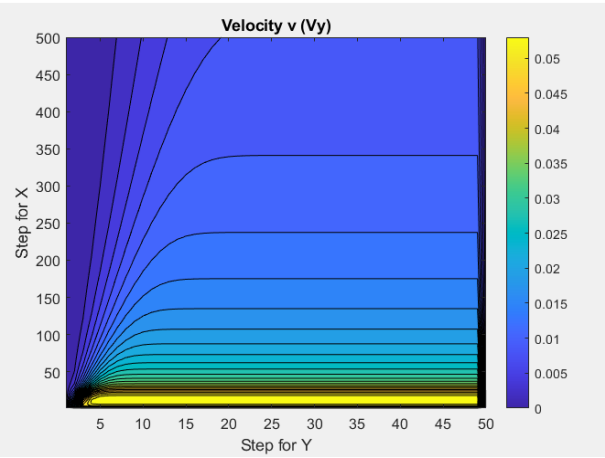


(a) Velocity Distribution Contour along x-direction

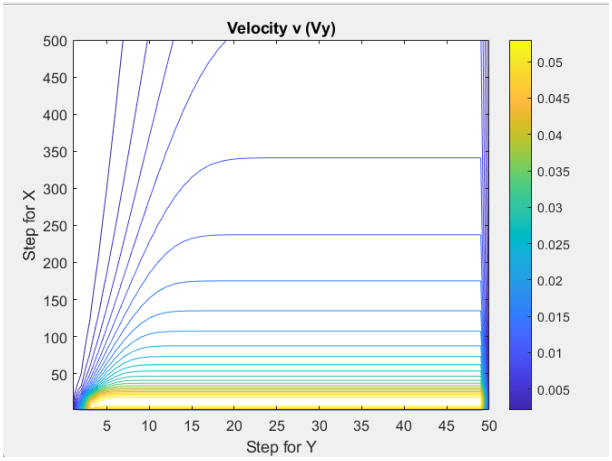


(b) Streamline Distribution Contour along x-direction

Figure 1: Velocity and Streamline Distribution Contours along x-direction corresponding to fluid flow over plate



(a) Velocity Distribution Contour along y-direction



(b) Streamline Distribution Contour along y-direction

Figure 2: Velocity and Streamline Distribution Contours along y-direction corresponding to fluid flow over plate

Here the solution strategy can be devised such that the Continuity equation is expressed as:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{15}$$

and the Navier Stokes can be expressed in 2D by simplification as:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}(\frac{\partial^2 u}{\partial y^2}) \tag{16}$$

and heat transfer equation can be expressed as:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial^2 y} \tag{17}$$

By the application of Finite Difference Approximation on both the Navier Stokes Equation and Continuity Equation, expressions can be related: By forward and central 1st order FDA on Navier Stokes results in:

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} \tag{18}$$

$$\frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} \tag{19}$$

$$\frac{\partial^2 u}{\partial^2 y} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} \tag{20}$$

The final form of discretized Navier Stokes Equation can be expressed as:

$$u_{i+1,j}(\frac{u_{i+1,j} - u_{i,j}}{\Delta x}) + v_{i,j}(\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}) = \frac{\mu}{\rho}(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2}) \tag{21}$$

or,

$$u_{i+1,j} = u_{i,j} + \frac{\mu}{\rho}(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2})\frac{\Delta x}{u_{i,j}} - (v_{i,j}\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y})\frac{\Delta x}{u_{i,j}} \tag{22}$$

Similarly for the continuity equation, discretization can be performed such that:

$$\frac{v_{i+1,j} - v_{i,j-1}}{\Delta y} = \frac{-1}{2}(\frac{u_{i+1,j-1} - u_{i,j-1}}{\Delta x} + \frac{u_{i+1,j} - u_{i,j}}{\Delta x}) \tag{23}$$

The final form of discretized continuity equation can be expressed as:

$$v_{i+1,j} = v_{i+1,j-1} - \frac{\Delta y}{2}((\frac{u_{i+1,j-1} - u_{i,j-1}}{\Delta x} + \frac{u_{i+1,j} - u_{i,j}}{\Delta x}) \tag{24}$$

The final form of discretized energy equation can be expressed as:

$$u_{i,j}(\frac{T_{i+1,j} - T_{i,j}}{\Delta x}) + v_{i,j}(\frac{T_{i,j+1} - T_{i,j-1}}{2\Delta y}) = \frac{\mu}{\rho}(\frac{T_{i,j+1} - 2T_{i,j} - T_{i,j-1}}{(\Delta y)^2}) \tag{25}$$

The equation which was used for evaluating the coefficient of friction is as follows Sanders (2014):

$$C_f = \frac{0.664}{Re^{\frac{1}{2}}} \tag{26}$$

```
    clc
clear
format long
%Defining of parameters
ui=7; %freestream velocity
Tin=25; %Inlet temperature
nu=1.6e-5; %kinematic viscosity forall nu=mu/rho
X=1; %Length of plate
Pr=0.71; %Prandtl Number
alpha=nu/Pr; %Thermal diffusivity
q=2100; %Constant Heat Flux
Re=(ui*X/nu); %Reynolds Number
k=0.0255; %Thermal conductivity
if Re>5e5
    disp('Flow is not Laminar rather Turbulent  hence causing inconsistency in
    caluclations')
end
Y=0.025; %Horizontal disance from plate
si=4.92*((Re)^(-0.5)); %Boundary Layer Height from end of plate
so=(si/Y)*100;          %Percentage of Boundary Layer Creation
%Generation of Analysis Grid
imax=500;
jmax=50;
dx=X/imax;
dy=Y/jmax;
h=28:(-6/imax):22;
%Generation of Matrix
u=ones(imax,jmax);
V=zeros(imax,jmax);
x=zeros(1,imax);
y=zeros(1,jmax);
T=ones(imax,jmax);
%Application of Boundary Conditions
u(:,:)=1;
u(:,1)=0;
u(1,:)=ui;
u(:,jmax)=ui;
u(1,1)=0;
V(1,:)=0;
V(:,1)=0;
V(1,1)=0;
T(1,:)=Tin;
T(:,jmax)=Tin;
```

```
%Navier Stokes and Continuity Equation
for o=1:imax
    for j=2:jmax-1
        for i=1:imax-1
            u(i+1,j)=(u(i,j))+(nu*(u(i,j+1)-2*u(i,j)+u(i,j-1))*(dx))/((u(i,j)*dy^2))-((u(
    i,j+1)-u(i,j-1))*V(i,j)*dx)/(2*u(i,j)*dy);

            V(i+1,j)=V(i+1,j-1)-((0.5*dy*(u(i+1,j)-u(i,j)+u(i+1,j-1)-u(i,j-1)))/dx);
        end
        end
        end
%Energy Equation
for m=1:imax
    for j=2:jmax-1
        for i=1:imax-1
            T(i,1)=(T(i,2)+q/h(1,i));

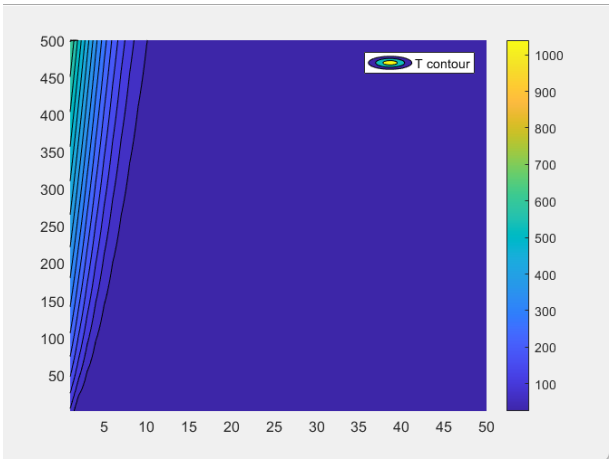
            T(i+1,j)=(T(i,j))+(alpha*(T(i,j+1)-2*T(i,j)+T(i,j-1))*(dx))/(u(i,j)*dy^2)+((T
    (i,j+1)-T(i,j-1))*(V(i,j)*dx)/u(i,j)*2*dy);

            T(imax,1)=(T(imax-1,2))+T(imax,2);

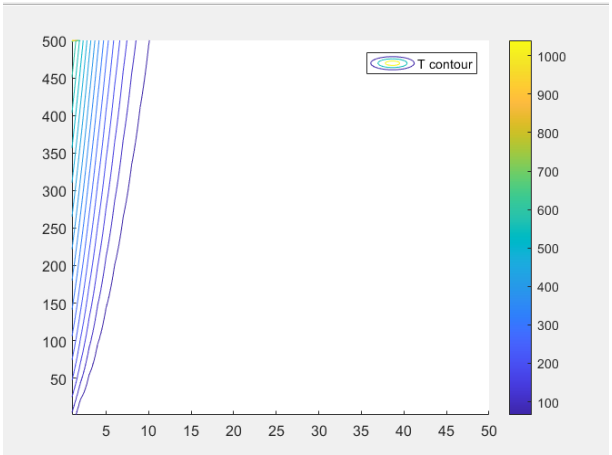
        end
        end
        end
%Calculation of Nusselts Number and Coefficient of Friction
i=50;
re1=(u(i,2)*(dx*(i))/nu);
Cfx1=0.664*re1^(-0.5);
Cfx=(2*nu*(u(i,2)-u(i,1)))/(dy*ui^2);
nu1=0.453*Pr^(1/3)*re1^(1/2);
Nu1=(q*(dx*i))./(k*T(i,1)-T(i,2));
%Minimum Value of Matrix u (Stability requirement check)
C=u(:,2);
A=abs(C);
A(isnan(A))=0;
B=nonzeros(A);
E=min(B);
%Stability Criterion
R=(0.5*E*dy^2)/nu;
dx;
%Checking stability
for i=1
    if R<dx
        disp('The Stability condition is declined by imax or jamx')
        disp("Evaluate for other values" )
            end
end
%Plotting and Displaying of Result

figure(1)
colorbar
hold on
contourf(u,25)
legend('u contour')
hold off
figure (2)
colorbar
hold on
contourf(T,25)
legend ('T contour')
hold off
figure (3)
colorbar hold on
contour(u,25)
legend ('u contour') hold off
figure (4)
colorbar hold on
contour(T,25)
legend ('T contour') hold off
```

Listing 2: Analysis of Flow over Plate using Navier Stokes and Continuity Equations (Constant Heat Flux)

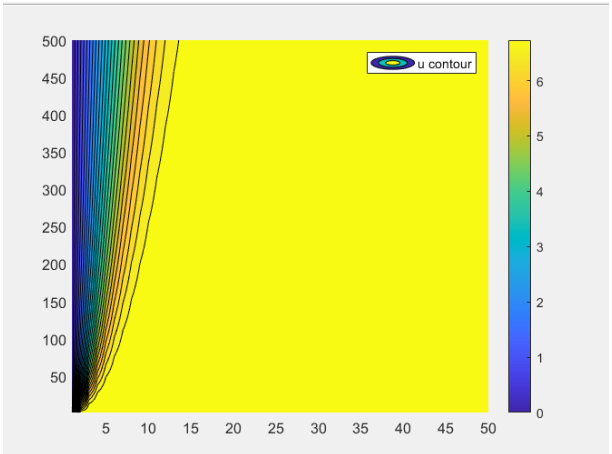


(a) Temperature Distribution Contour along plate

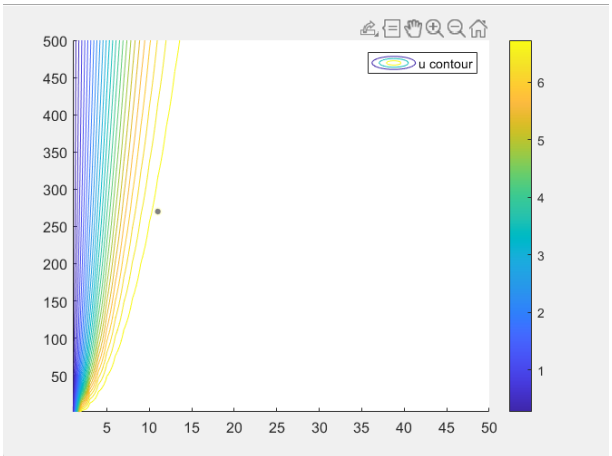


(b) Streamline Distribution Contour along plate

Figure 3: Temperature and Streamline Distribution Contours corresponding to fluid flow over plate



(a) Velocity Distribution Contour along plate



(b) Streamline Distribution Contour along plate

Figure 4: Velocity and Streamline Distribution Contours corresponding to fluid flow over plate

4 Conclusion

The analysis of fluid flow over a plate using the Navier Stokes and Continuity equations, both with and without heat flux, provides valuable insights into the fundamental principles governing boundary layer behavior, velocity distributions, and thermal effects. By applying finite difference approximations, the discretized forms of these governing equations allow for numerical solutions that predict velocity and temperature distributions with reasonable accuracy. The results illustrate the influence of viscosity and heat transfer on flow stability, demonstrating the importance of factors such as boundary layer development, no-slip conditions, and thermal diffusion. The velocity contours and streamline distributions highlight the impact of different flow parameters on the overall fluid motion, while the inclusion of heat flux reveals its effect on temperature gradients and thermal boundary layers. The study also emphasizes the significance of stability criteria in computational modeling to ensure reliable results. Ultimately, this work contributes to the broader understanding of fluid mechanics and heat transfer, offering a computational approach to solving real-world engineering problems related to aerodynamics, thermal management, and industrial fluid systems. Future research could explore turbulence effects, variable viscosity, or three-dimensional simulations for a more comprehensive analysis of complex fluid behaviors.The contour plots of velocity and temperature distributions further reveal critical flow characteristics. The velocity contour maps depict the development of a boundary layer, with higher velocity gradients near the plate's surface and a gradual transition to free-stream velocity as the distance from the plate increases. This behavior is consistent with the classic laminar boundary layer theory, where shear stress is most pronounced near the surface due to the no-slip condition. The streamline distributions provide a visualization of flow directionality, emphasizing the smooth transition of velocity along the plate. When heat flux is introduced, the temperature contours exhibit a clear thermal boundary layer formation, where temperature gradients are steepest near the plate. The interaction between velocity and temperature distributions demonstrates the coupled nature of convective heat transfer, with

regions of higher velocity promoting increased thermal diffusion. The contours also help in identifying thermal boundary layer thickness, which depends on the Prandtl number and Reynolds number of the flow. Additionally, the stability of the numerical scheme is evident in the smoothness and coherence of the contour plots, which ensure that the discretized equations properly capture the expected physical behavior. The study also emphasizes the importance of selecting appropriate grid resolutions and stability criteria to prevent computational errors or artificial oscillations in the results.

References

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