Similarity Solution for Uniform Wall Temperature Corresponding to High-Prandtl Number Viscous Incompressible Parallel Fluid Flow over a Semi-Infinite Flat Plate

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Abstract

This work presents an analytical similarity solution for laminar, steady, viscous incompressible flow of high-Prandtl-number fluids over an isothermal semi-infinite flat plate. Representative of oils and other highly viscous heat transfer media, such flows are characterized by a thermal boundary layer embedded within a much thicker velocity layer. By introducing an appropriate similarity variable and approximating the velocity distribution as linear within the thermal region, the governing energy equation is reduced to an ordinary differential equation. The resulting solution, expressed in integral form and evaluated numerically, yields the complete temperature profile and enables direct determination of the local and average Nusselt numbers. The analysis shows that for high-Pr fluids, the scaling factor is $Pr^{1/3}$, and the derived coefficients (0.3387 for local, 0.678 for average Nusselt number) align with established theoretical predictions. This provides engineers with an efficient analytical tool for accurate heat transfer estimation in high-Pr applications such as lubricating systems, polymer processing, and thermal management of high-viscosity fluids.

The idea of similarity transformation is to transform a complex partial differential equation to an ordinary differential equations by combining the variables in a special manner. The transformation by similarity variable is typically successful provided the similarity variable remains an independent variable. Through transformation, the number of independent variables are reduced by one, and hence for said reason, two of the original boundary conditions reduce to one.

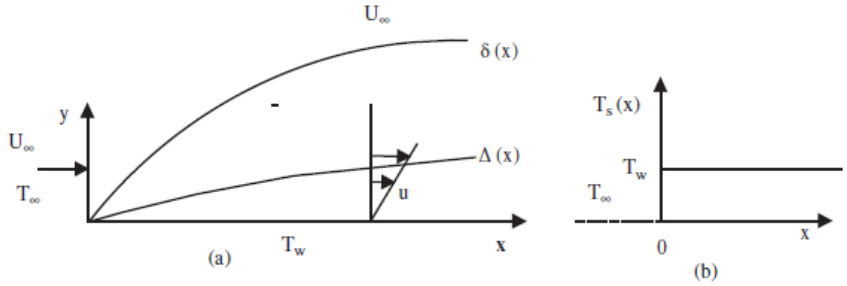


Figure 1: Velocity and thermal boundary layer over a flat plate for Pr>> 1 Onur (2023)

Consider 1, when the fluid flows over an isothermal semi-infinite flat plate set at zero angle of incidence to a uniform freestream velocity U_∞. The plate is maintained at a constant temperature T_w, as shown in 1. Freestream temperature T_∞ is regarded constant, and the body forces are neglected for ease in analysis. Since the flow is assumed for low velocity, viscous dissipation is also ignored and the flow is assumed to be viscous incompressible, steady-state, and two-dimensional along with constant fluid properties. Considering constant fluid properties implies the velocity distribution being independent of temperature distribution. This means that the energy and momentum equations are not coupled together. Further assuming that the temperature difference between wall temperature T_w and freestream temperature T_∞ is small, thus variation of fluidic properties with temperature is ignored. Reynolds number is high enough for allowing the validity of boundary layer assumption. This problem is consistent with the momentum

transfer problem by Blasius. The analysis is concerned with determining key properties such as boundary layer thickness, surface heat flux, heat transfer coefficient, and Nusselt number.
For high-Prandtl-number fluids such as oils, the velocity boundary layer is comparatively much thicker than the thermal boundary layer ($\delta << \Delta$) which can be implied from 1. This means that the thermal boundary layer is embedded within the momentum boundary layer. Consider the energy equation

$$\frac{d^2\theta}{d\eta^2} + \frac{Pr}{2} f(\eta) \frac{d\theta}{d\eta} = 0 \tag{1}$$

where $\theta = \frac{T-T_w}{T_\infty - T_w}$ and $\eta = \frac{y}{\sqrt{\frac{U_\infty}{\nu x}}}$. A strong approximation for the dimensionless velocity profile for high values of Prandtl number fluids is to assume that the velocity profile is linear throughout the thermal boundary layer, with the slope given by the value evaluated at the wall.
For $\eta = 0$, the approximation implies

$$\frac{d^2 f}{d\eta^2} = 0.332 \tag{2}$$

Let $A = 0.332$, the 1 can be integrated twice

$$f = \frac{A}{2}\eta^2 + C_1\eta + C_2 \tag{3}$$

The boundary conditions are as

$$f(0) = 0$$
$$(\frac{df}{d\eta})_{\eta=0} = 0$$

Using the above conditions, the integration constants C_1 and $], C_2$ produce null value Thus

$$f = \frac{A}{2}\eta^2 \tag{4}$$

Substituting the dimensionless velocity expression in **??**, the following equation is obtained

$$\frac{d^2\theta}{d\eta^2} + (0.332Pr)\left(\frac{\eta^2}{4}\right)\frac{d\theta}{d\eta} = 0 \tag{5}$$

The boundary conditions are

$$\theta(0) = 0$$
$$\theta(\infty) = 1$$

Let $C = 0.3321Pr$ and $p = \frac{d\theta}{d\eta}$, then the differential equation becomes

$$\frac{dp}{d\eta} + C\frac{\eta^2}{4}p = 0 \tag{6}$$

Through separation of variables

$$p = Bexp\left(-\frac{C}{12}\eta^3\right) \tag{7}$$

Introducing $p = \frac{d\theta}{d\eta}$ and integrating once again with respect to η results in the following expression

$$\theta = B\int_0^\eta exp\left(-\frac{C}{12}\eta^3\right)d\eta + D \tag{8}$$

Using the boundary condition $\theta(0) = 0$ results in D=0

$$\theta = B\int_0^\eta exp\left(-\frac{C}{12}\eta^3\right)d\eta \tag{9}$$

Using the boundary condition $\theta(\infty) = 1$ gives the unknown constant B as

$$B = \frac{1}{\int_0^\infty exp\left(-\frac{C}{12}\eta^3\right)d\eta} \tag{10}$$

The temperature distribution finally takes shape as

$$\frac{\int_0^{\eta} exp\left(-\frac{C}{12}\eta^3\right) d\eta}{\int_0^{\infty} exp\left(-\frac{C}{12}\eta^3\right) d\eta} \tag{11}$$

> *restart*;
>
> *assume*$(C, positive)$;
> $\int_0^{\infty} \exp(-\frac{C}{12} \cdot \eta^3)\, d\eta$;

$$\frac{2\,12^{\frac{1}{3}}\pi\sqrt{3}}{9C^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} \tag{12}$$

> Inserting $C = 0.3321Pr$ into yhe result and evaluating the expression gives the following value

$$\int_0^{\infty} exp\left(-\frac{C}{12}\eta^3\right) d\eta = \frac{2.9521}{Pr^{1/3}} \tag{13}$$

The temperature distribution can finally be obtained as

$$\theta = 0.3387Pr^{1/3} \int_0^{\eta} exp\left(-\frac{Pr\eta^3}{36.1336}\right) d\eta \tag{14}$$

Considering any value of Prandtl number, the integral can be evaluated numerically. The local Nusselt number has to be evaluated

$$Nu_x = \frac{xh}{k} = \frac{xq_w^{"}}{k(T_w - T_\infty)} = \frac{-x\left(\frac{\partial T}{\partial y}\right)_{y=0}}{(T_w - T_\infty)} \tag{15}$$

$$T = T_w + (T_\infty - T_w)\theta \tag{16}$$

$$\frac{\partial T}{\partial y} = (T_\infty - T_w)\left(\frac{d\theta}{d\eta}\right)_{\eta=0}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial T}{\partial y} = (T_\infty - T_w)\frac{1}{x}\sqrt{Re_x}\left(\frac{d\theta}{d\eta}\right)_{\eta=0} \tag{17}$$

Hence

$$Nu_x = \sqrt{Re_x}\left(\frac{d\theta}{d\eta}\right)_{\eta=0} \tag{18}$$

and $\left(\frac{d\theta}{d\eta}\right)_{\eta=0}$ can be obtained from temperature solution $\theta(\eta)$ as

$$\left(\frac{d\theta}{d\eta}\right)_{\eta=0} = 0.3387Pr^{1/3} \tag{19}$$

The local Nusselt number becomes

$$Nu_x = 0.3387Pr^{1/3}\sqrt{Re_x} \tag{20}$$

The average Nusselt number is

$$\overline{Nu_L} = 0.678\sqrt{Re_L}Pr^{1/3} \tag{21}$$

Conclusion

This study demonstrates that the similarity transformation method, coupled with high-Pr velocity profile approximations, offers a powerful means of predicting heat transfer behavior in viscous fluids where momentum diffusion dominates over thermal diffusion. The thermal boundary layer was shown to be significantly thinner than the velocity boundary layer, a defining characteristic of high-Pr flows. Analytical expressions for the temperature distribution, local Nusselt number, and average Nusselt number were derived, with heat transfer scaling as $Pr^{1/3}$. The obtained constants closely match benchmark correlations, confirming the model's accuracy and practical utility. The approach is computationally efficient, physically transparent, and well-suited for preliminary design and optimization of systems involving high-viscosity, low-thermal-diffusivity fluids.

References

Onur, N. (2023). *Introduction to convective heat transfer a software-based approach using maple and matlab*. Wiley: John Wiley Sons. DOI: 9781119766766