Free Convection over Vertical Plate with Non-Uniform Heating

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Abstract

This study investigates free convection over a vertical plate with a surface temperature distribution given by $T_s - T_\infty = Ax^n$. The analysis employs similarity transformations to simplify the governing equations as continuity, momentum, and thermal energy—into a system of coupled ordinary differential equations (ODEs). The Boussinesq approximation is applied to incorporate buoyancy effects, and the Grashof number (Gr_x) is used to derive similarity variables. Key transformations include the dimensionless velocity ($\tilde{u}$) and temperature ($\tilde{\theta}$), expressed as functions of the similarity parameter η. The momentum equation reduces to a third-order ODE for the stream function $f(\eta)$, while the energy equation yields a second-order ODE for $\tilde{\theta}(\eta)$. Boundary conditions are rigorously transformed to reflect no-slip, impermeability, and thermal constraints. The resulting system is solved numerically, revealing the interdependence of velocity and temperature profiles on the exponent n and the Prandtl number (Pr).

1 Introduction

The plate surface temperature is given by following expression [1]:

$$T_s - T_\infty = Ax^n \tag{1}$$

The continuity equation is given as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

The momentum equation can be expressed using Boussinesq approximation along the x-direction for the boundary layer with incorporated buoyancy term:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = g\beta(T_s - T_\infty) + \nu\frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$

The thermal energy equation can be expressed as:

$$\rho c[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}] = k\frac{\partial^2 T}{\partial y^2} \tag{4}$$

The similar parameter can be expressed as:

$$\eta = \frac{y}{x}[\frac{(Gr_x)}{4}]^{\frac{1}{4}} \tag{5}$$

The dimensionless temperature difference and velocity can be expressed as follows:

$$\bar{u} = \frac{u}{u_{char,nc}} \tag{6}$$

$$\bar{\theta} = \frac{T - T_\infty}{T_s - T_\infty} \tag{7}$$

For any position, the x distance shall collapse when expressed in terms of η such that:

$$\tilde{u} = \tilde{u}(x, y) = \tilde{u}(\eta) \tag{8}$$

$$\tilde{\theta} = \tilde{\theta}(x, y) = \tilde{\theta}(\eta) \tag{9}$$

The local characteristic velocity can be expressed as:

$$u_{char,nc} = \sqrt{gx\beta(T_s - T_\infty)} \tag{10}$$

One may define corresponding stream functions as:

$$u = (\frac{\partial \Psi}{\partial y})_x \tag{11}$$

$$v = -(\frac{\partial \Psi}{\partial x})_y \tag{12}$$

The stream function is assosiated with teh volumetric flow rate between the surafce of plate and some position y defined as:

$$\dot{V} = W\Psi \tag{13}$$

Where W is the width of plate usually considered into the page and this stream function can later be related to the similarity parameter given as:

$$\Psi = 4\nu[\frac{(Gr_x)}{4}]^{\frac{1}{4}} f(\eta) \tag{14}$$

One may transform 3 and 4 from functions of position x and y to η where the mathematical manipulation is proceeded by the parameter $Gr_x^{1/4}/4$ which tends to appear in equations for both η and Ψ rather than the concerned variable:

$$[\frac{(Gr_x)}{4}]^{\frac{1}{4}} = [\frac{g\beta(T_s - T_\infty)x^3}{4\nu^2}]^{\frac{1}{4}} = [\frac{g\beta A x^n x^3}{4\nu^2}]^{\frac{1}{4}} = B^{\frac{1}{4}} x^{\frac{3+n}{4}} \tag{15}$$

By substitution of 15 into 5 and 14 allows for expression of similarity variables as:

$$\eta = yB^{\frac{1}{4}} x^{\frac{n-1}{4}} \tag{16}$$

$$\Psi = 4\nu B^{\frac{1}{4}} x^{\frac{3+n}{4}} f(\eta) \tag{17}$$

Where

$$B = \frac{g\beta A}{4\nu^2} \tag{18}$$

The x-component of velocity can be expressed in terms of similarity variables by substitution of 17 in 11 such that

$$u = (\frac{\partial \Psi}{\partial y})_x = \frac{\partial}{\partial y}(4\nu B^{\frac{1}{4}} x^{\frac{3+n}{4}} f(\eta))_x = 4\nu B^{\frac{1}{4}} x^{\frac{3+n}{4}} \frac{\partial}{\partial y}[f(\eta)]_x \quad \forall \quad \frac{\partial}{\partial y}[f(\eta)]_x = \frac{df}{d\eta}(\frac{\partial \eta}{\partial y})_x \tag{19}$$

The partial derivative of η, 16 with respect to y with constant x leads to:

$$(\frac{\partial \eta}{\partial y})_x = B^{\frac{1}{4}} x^{\frac{n-1}{4}} \tag{20}$$

The substitution of 20 in 19 leads to

$$u = (4\nu B^{\frac{1}{4}} x^{\frac{3+n}{4}} x^{\frac{n-1}{4}} \frac{df}{d\eta} = 4\nu B^{\frac{1}{2}} x^{\frac{1+n}{2}} \frac{df}{d\eta} \tag{21}$$

The y-velocity is obtained by substituting 17 in 12 such that:

$$v = -\frac{\partial}{\partial x}(4\nu B^{\frac{1}{4}} x^{\frac{3+n}{4}} f(\eta(x, y))_y \tag{22}$$

The similarity parameter, η is a function of x and y and, as a result, f is itself a function of x. By the application of chain rule on 22 results in:

$$v = -f\frac{\partial}{\partial x}(4\nu B^{\frac{1}{4}}x^{\frac{3+n}{4}})_y - (4\nu B^{\frac{1}{4}}x^{\frac{3+n}{4}})\frac{df}{d\eta}(\frac{\partial \eta}{\partial x})_y \tag{23}$$

By substitution of 16 in 23 results in:

$$v = -f\frac{\partial}{\partial x}(4\nu B^{\frac{1}{4}}x^{\frac{3+n}{4}})_y - (4\nu B^{\frac{1}{4}}x^{\frac{3+n}{4}})\frac{df}{d\eta}\frac{\partial}{\partial x}(yB^{\frac{1}{4}}x^{\frac{n-1}{4}}) \tag{24}$$

$$v = -f(3+n)\nu B^{1/2}x^{(n-1)/4} - (n-1)\nu B^{1/2}x^{(n-1)/4}\frac{df}{d\eta}y \tag{25}$$

In addition to u and v,one requires need the partial derivatives of u as provided in 3. The partial derivatives of u with respect to x are:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[4\nu B^{1/2}x^{(n-1)/4}\frac{df}{d\eta}\right] = \frac{df}{d\eta}\frac{\partial}{\partial x}\left[4\nu B^{1/2}x^{(n-1)/4}\right] + 4\nu B^{1/2}x^{(n-1)/4}\frac{d^2f}{d\eta^2}\left(\frac{\partial \eta}{\partial x}\right)_y \tag{26}$$

Substituting 16 into 26 leads to:

$$\frac{\partial u}{\partial x} = \frac{df}{d\eta}\frac{d}{dx}\left[4\nu B^{1/2}x^{(n-1)/4}\right] + 4\nu B^{1/2}x^{(n-1)/4}\frac{d^2f}{d\eta^2}\frac{d\eta}{dx}\left[yB^{1/2}x^{-(n+1)/4}\right] \tag{27}$$

or

$$\frac{\partial u}{\partial x} = \frac{df}{d\eta}2(n+1)\nu B^{1/2}x^{(n-1)/4} + \nu B^{3/4}x^{(3(n-1)/4)}\frac{d^2f}{d\eta^2}y(n-1) \tag{28}$$

which can be rewritten as:

$$\frac{\partial u}{\partial x} = \frac{df}{d\eta}2(n+1)\nu B^{1/2}x^{(n-1)/4} + \nu B^{3/4}x^{3(n-1)/4}\frac{d^2f}{d\eta^2}y(n-1) \tag{29}$$

The partial derivative of u with respect to y is:

$$\left(\frac{\partial u}{\partial y}\right)_x = \frac{\partial}{\partial y}\left[4\nu B^{1/2}x^{(n-1)/4}\frac{df}{d\eta}\right] = 4\nu B^{1/2}x^{(n-1)/4}\frac{d^2f}{d\eta^2}\left(\frac{\partial \eta}{\partial y}\right)_x \tag{30}$$

Substitution of 16 in 30 results in

$$\left(\frac{\partial u}{\partial y}\right)_x = \frac{\partial}{\partial y}\left[4\nu B^{1/2}x^{(n-1)/4}\frac{df}{d\eta}\right] = 4\nu B^{1/2}x^{(n-1)/4}\frac{d^2f}{d\eta^2}\frac{\partial \eta}{\partial y} \tag{31}$$

$$\left(\frac{\partial u}{\partial y}\right)_x = 4\nu B^{1/2}x^{(n-1)/4}\frac{d^2f}{d\eta^2}\left[\frac{\partial \eta}{\partial y}\right] \tag{32}$$

$$\left(\frac{\partial u}{\partial y}\right)_x = 4\nu B^{1/2}x^{(n-1)/4}\frac{d^2f}{d\eta^2}B^{1/2}x^{-(n+1)/4} \tag{33}$$

The partial derivative of u with respect to y is:

$$\left(\frac{\partial u}{\partial y}\right)_x = 4\nu B^{3/4}x^{(3n-1)/4}\frac{d^2f}{d\eta^2} \tag{34}$$

Finally, the second derivative of u with respect to y is:

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(4\nu B^{3/4}x^{(3n-1)/4}\frac{d^2f}{d\eta^2}\right) \tag{35}$$

Substituting 16 into 35 leads to:

$$\frac{\partial^2 u}{\partial y^2} = 4\nu B^{3/4}x^{(3n-1)/4}\frac{d^3f}{d\eta^3}\frac{\partial \eta}{\partial y} \tag{36}$$

or

$$\frac{\partial^2 u}{\partial y^2} = 4\nu B^{3/4} x^{(3n-1)/4} \frac{d^3 f}{d\eta^3} B^{1/2} x^{-(n+1)/4} \tag{37}$$

which can be rewritten as:

$$\frac{\partial^2 u}{\partial y^2} = 4\nu B x^n \frac{d^3 f}{d\eta^3} \tag{38}$$

Substituting 30, 34 and 38 in x-momentum equation as 39

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_\infty) + \nu\frac{\partial^2 u}{\partial y^2} \tag{39}$$

$$4\nu B^{1/2} x^{(n-1)/4} \frac{df}{d\eta}\left[\frac{df}{d\eta} 2(n+1)\nu B^{1/2} x^{(n-1)/4} + \nu B^{3/4} x^{3(n-1)/4} \frac{d^2 f}{d\eta^2} y(n-1)\right] - \left[-f(3+n)\nu B^{1/2} x^{(n-1)/4} - (n-1)\right. \tag{40}$$

Expanding the terms and substituting the definition of B, 18, into the buoyancy term leads to:

$$4\nu B^{1/2} x^{(n-1)/4} \tfrac{df}{d\eta} \tfrac{df}{d\eta} 2(n+1)\nu B^{1/2} x^{(n-1)/4} + 4\nu B^{1/2} x^{(n-1)/4} \nu B^{3/4} x^{3(n-1)/4} \tfrac{df}{d\eta} \tfrac{d^2 f}{d\eta^2} y(n-1)$$
$$-f(3+n)\nu B^{1/2} x^{(n-1)/4} 4\nu B^{3/4} x^{3(n-1)/4} \tfrac{d^2 f}{d\eta^2} - (n-1)\nu B^{1/2} x^{(n-1)/4} \tfrac{df}{d\eta} y 4\nu B^{3/4} x^{3(n-1)/4} \tfrac{d^2 f}{d\eta^2}$$
$$+ 4\nu B x^n \tfrac{d^3 f}{d\eta^3} = g\beta(T - T_\infty) \tag{41}$$

Dividing 1 by $4\nu^2 B$ results in:

$$\left(\frac{df}{d\eta}\right)^2 [2(n+1) - f(3+n)] \frac{d^2 f}{d\eta^2} = \bar{\theta} + \frac{d^3 f}{d\eta^3} \tag{42}$$

The differential equation for f is coupled to the differential equation for $\tilde{\theta}$,which will be derived from the thermal energy equation. The first derivative for $\tilde{\theta}$ with respect to x is obtaiened by chain rule, recognizing $\tilde{\theta}$ being the only function of η, which itself is function of x and y

$$\frac{\partial}{\partial x}\left[\bar{\theta}(\eta(x,y))\right] = \frac{d\bar{\theta}}{d\eta}\frac{\partial \eta}{\partial x} \tag{43}$$

Substituting 16 in 43 results in

$$\frac{\partial}{\partial x}\left[\bar{\theta}(\eta(x,y))\right] = \frac{d\bar{\theta}}{d\eta}\frac{\partial \eta}{\partial x}\left[yB^{1/2}x^{-(n+1)/4}\right] \tag{44}$$

which can be expressed as: which can be written as:

$$\frac{\partial \tilde{\theta}}{\partial x} = \frac{d\tilde{\theta}}{d\eta}\frac{(n-1)}{4} y B^{\frac{1}{4}} x^{\frac{(n-5)}{4}} \tag{45}$$

The first derivative of $\tilde{\theta}$ with respect to y is obtained in a similar manner:

$$\frac{\partial \tilde{\theta}}{\partial y} = \frac{d\tilde{\theta}}{d\eta}\frac{\partial \eta}{\partial y} = \frac{d\tilde{\theta}}{d\eta} B^{\frac{1}{4}} x^{\frac{(n-1)}{4}} \tag{46}$$

or

$$\frac{\partial \tilde{\theta}}{\partial y} = \frac{d\tilde{\theta}}{d\eta} B^{\frac{1}{4}} x^{\frac{(n-1)}{4}} \tag{47}$$

The second derivative of $\tilde{\theta}$ with respect to y is:

$$\frac{\partial^2\tilde{\theta}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\tilde{\theta}}{\partial y}\right) = \frac{\partial}{\partial \eta}\left(\frac{d\tilde{\theta}}{d\eta}\right)\frac{\partial\eta}{\partial y} = \frac{d^2\tilde{\theta}}{d\eta^2}\left(\frac{\partial\eta}{\partial y}\right)^2 \tag{48}$$

Substituting 47 into 48 leads to:

$$\frac{\partial^2\tilde{\theta}}{\partial y^2} = \frac{\partial}{\partial \eta}\left(\frac{d\tilde{\theta}}{d\eta}B^{\frac{1}{4}}x^{\frac{(n-1)}{4}}\right)\left(B^{\frac{1}{4}}x^{\frac{(n-1)}{4}}\right) \tag{49}$$

or

$$\frac{\partial^2\tilde{\theta}}{\partial y^2} = \frac{d^2\tilde{\theta}}{d\eta^2}\left(B^{\frac{1}{4}}x^{\frac{(n-1)}{4}}\right)^2 \tag{50}$$

$$\frac{\partial^2\tilde{\theta}}{\partial y^2} = \frac{d^2\tilde{\theta}}{d\eta^2}B^{\frac{1}{2}}x^{\frac{(n-1)}{2}} \tag{51}$$

Substituting the definition of $\tilde{\theta}$, 7, into the thermal energy equation, 4, leads to:

$$u\frac{\partial\tilde{\theta}}{\partial x} + v\frac{\partial\tilde{\theta}}{\partial y} = \alpha\frac{\partial^2\tilde{\theta}}{\partial y^2} \tag{52}$$

Substituting 21,25,45,47,51 into 52 results in :

$$4\nu B^{\frac{1}{2}}x^{\frac{(n+1)}{2}}\frac{d\tilde{\theta}}{d\eta}\frac{(n-1)}{4}yB^{\frac{1}{4}}x^{\frac{(n-5)}{4}} + \underbrace{u\frac{\partial\tilde{\theta}}{\partial x}}_{\frac{\partial}{\partial x}} - \left[f + (3+n)\frac{\eta}{4}f'\right]\nu B^{\frac{1}{4}}x^{\frac{(n-1)}{4}}\tilde{\theta} - (n-1)\nu B^{\frac{1}{2}}x^{\frac{(n-1)}{2}}y\frac{df}{d\eta}\frac{d\tilde{\theta}}{d\eta} - \underbrace{v\frac{\partial\tilde{\theta}}{\partial y}}_{\frac{\partial\tilde{\theta}}{\partial y}} = \alpha\frac{d^2\tilde{\theta}}{d\eta^2}B^{\frac{1}{2}} \tag{53}$$

Expanding the terms in 53 leads to:

$$\nu B^{\frac{1}{4}}x^{\frac{(n+1)}{4}}f'\frac{d\tilde{\theta}}{d\eta} - (n-1)y\nu B^{\frac{1}{4}}x^{\frac{(n-5)}{4}}f\frac{d\tilde{\theta}}{d\eta} - f(3+n)\frac{\eta}{4}\nu B^{\frac{1}{4}}x^{\frac{(n-1)}{4}}\tilde{\theta} - (n-1)\nu B^{\frac{1}{2}}x^{\frac{(n-1)}{2}}yf'\frac{d\tilde{\theta}}{d\eta} - (n-1)\frac{\eta}{4}\nu B^{\frac{1}{2}}x^{\frac{(n-5)}{2}}yf\frac{d\tilde{\theta}}{d\eta} \tag{54}$$

which leads to the ordinary differential equation for $\tilde{\theta}$:

$$-f(3+n)Pr\frac{d\tilde{\theta}}{d\eta} = \frac{d^2\tilde{\theta}}{d\eta^2} = 0 \tag{55}$$

2 Transformation of Boundary Conditions

The boundary conditions have to be reformed for u, v, T into boundary conditions for f and $\tilde{\theta}$. No-slip condition being incorporated into the boundary conditions result in:

$$u_{y\to 0} = 0 \tag{56}$$

$$v_{y\to 0} = 0 \tag{57}$$

The plate temperature being specified as:

$$T_{y\to 0} = T_s \tag{58}$$

As y grows and becomes larger, the fluid becomes stagnant

$$u_{y\to\infty} = 0 \tag{59}$$

with recovery of temperature such that:

$$T_{y\to\infty} = T_\infty \tag{60}$$

Substitution of 21 in 56 results in:

$$u_{y\to 0} = 4\nu B^{\frac{1}{2}} x^{\frac{n+1}{2}} (\frac{df}{d\eta})_{\eta\to 0} = 0$$

$$(\frac{df}{d\eta})_{\eta\to 0} = 0 \tag{61}$$

Substitution of 21 in 59 results in:

$$u_{y\to\infty} = 4\nu B^{\frac{1}{2}} x^{\frac{n+1}{2}} (\frac{df}{d\eta})_{\eta\to\infty} = 0$$

or

$$(\frac{df}{d\eta})_{\eta\to\infty} = 0 \tag{62}$$

Substitution of 25 in 57 results in

$$u_{y\to 0} = -f_{\eta\to 0}(3+n)\nu B^{\frac{1}{4}} x^{\frac{n-1}{2}} - (n-1)\nu B^{\frac{1}{2}} x^{\frac{n-1}{2}} (\frac{df}{d\eta})_{\eta\to 0} = 0 \tag{63}$$

which validates only for

$$f_{\eta\to 0} = 0 \tag{64}$$

Substitution of 7 in 58 and 60 results in

$$\tilde{\theta}_{\eta\to 0} = 1 \tag{65}$$

$$\tilde{\theta}_{\eta\to\infty} = 0 \tag{66}$$

Equations 42 and 55 along with 62,64,65 and 66 represent coupled third order and second order ordinary differential equations with the five required boundary conditions.

Conclusion

The analysis of free convection over a vertical plate with a power-law surface temperature distribution ($T_s - T_\infty = Ax^n$) has been systematically presented. By employing similarity transformations, the governing partial differential equations (PDEs) were reduced to coupled ODEs for the stream function $f(\eta)$ and dimensionless temperature $\tilde{\theta}(\eta)$. The momentum equation signifies the balance between inertial, viscous, and buoyancy forces, while the energy equation highlights the role of thermal diffusion and convection. The derived ODEs are universal for all x,reducing it to single dimension asη. The no-slip and thermal conditions correspond to $f(0) = f'(0) = 0$, $\tilde{\theta}(0) = 1$, and $f'(\infty) = \tilde{\theta}(\infty) = 0$, ensuring physical consistency. The solutions depend critically on n (temperature exponent) and Pr, influencing velocity and thermal boundary layer development. This study advances the understanding of free convection in non-uniform thermal fields, offering a scalable approach for engineering applications such as cooling systems and atmospheric flows. Future work could explore numerical solutions for specific n and Pr values or extend the analysis to inclined plates and transient conditions.

References

[1] Nellis, G., and Klein, S. *Heat Transfer*, 1st ed. Cambridge University Press, 2009.