Analytical Solution Analysis of Heat Transfer through Flux Meter

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Abstract

This study presents a comprehensive analysis of heat transfer through a flux meter, using analytical methods. The governing differential equation for temperature distribution within the plate was derived, considering radiative heat loss and contact resistance at the boundaries. Analytical solutions were obtained by solving the differential equation, while numerical validation was performed using Engineering Equation Solver (EES) and Maple. The temperature profile across the plate was plotted, and a calibration curve correlating heat flux to the temperature difference between the plate's center and casing was developed. Additionally, the uncertainty in heat flux measurement due to temperature measurement errors was quantified. The results demonstrate the efficacy of the proposed model in predicting heat flux and highlight the robustness of the analytical and numerical approaches.

1 Introduction

1.1 Problem Statement

A flux meter is illustrated in 3. A thin plate is clamped on either end to a casing that is maintained at $T_c = 20^o C$. There is a contact resistance between the plate and the casing, $R^{"} = 1 * 10^{-4} K m^2/W$. The thickness of theplate is $th = 1$ mm and the half-width of the plate is L = 2 cm.The plate conductivity is $k = 10 W/mK$. The back of the plate is insulated and the front of the plate is mounted within an evacuated enclosure in order to eliminate convection. You may model radiation loss from the plate surface using an effective "radiation" heat transfer coefficient, calculated according to $\bar{h}_{rad} = 4\varepsilon\sigma\bar{T}^3$, where $\bar{T}$ is the average absolute temperature of the plate and the surroundings, considered 300 K for this problem. The plate radiates to surroundings at $T_\infty = 20^o C$. The nominal flux on the plate is $\dot{q}^{''} = 900 W/m^2$. One may assume that the plate temperature distribution is 1-D in the x-direction. You may also assume that the emissivity of the plate, ε, isone and therefore all of the flux on the plate is absorbed. The flux meter operates by correlating the difference between the temperature at the center of the plate and the casing with the applied heat flux [1].

1. Derive the governing differential equation that governs the temperature within the plate.
2. What are the boundary conditions for the differential equation
3. Determine the solution to the differential equation from (1) subject to the boundary conditions from (2) without using Maple.
4. Plot the temperature in the plate as a function of position
5. Use Maple to solve the differential equation and obtain symbolic expressions for the boundary conditions. Implement the Maple expressions into EES and show that your answer is identical to the one determined in (3).
6. Prepare a calibration curve for the flux meter - plot the heat flux as a function of the difference between the temperature at the center of the plate and the casing.
7. If the uncertainty in the measurement of the temperature difference is $\delta\Delta T = 0.5$ K then what is the uncertainty in the measurement of the heat flux?

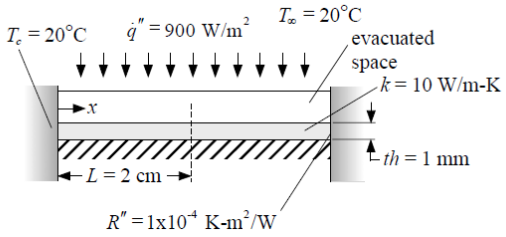


Figure 1: Flux meter

1.2 Solution

By the application of energy balance on the control volume as shown below 1, the following equation is the result

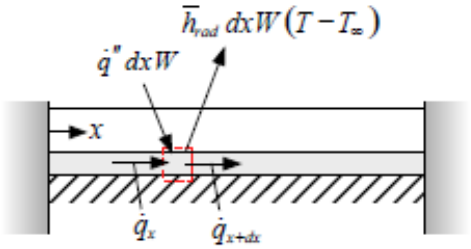


Figure 2: Differential energy balance

$$\dot{q}_x + \dot{q}^{"} dxW = \dot{q}_{x+dx} + \bar{h}_{rad} dxW(T - T_\infty) \tag{1}$$

The expansion leads to following expression:

$$\dot{q}_x + \dot{q}^{"} dxW = \dot{q}_x + \frac{d\dot{q}}{dx} dx + \bar{h}_{rad} dxW(T - T_\infty) \tag{2}$$

By the substitution of Fourier's Law, the above equation results as:

$$\dot{q}^{"} dxW = \frac{d}{dx}(-kWth\frac{dT}{dx})dx + \bar{h}_{rad} dxW(T - T_\infty) \tag{3}$$

or

$$\frac{d^2T}{dx^2} - \frac{\bar{h}_{rad}}{kth}T = \frac{-\dot{q}^{"}}{kth} - \frac{\bar{h}_{rad}}{kth}T_\infty \tag{4}$$

By application of interface energy balance at x=0 leads to:

$$\frac{(T_c - T_{x=0})}{R} = -k(\frac{dT}{dx})_{x=0} \tag{5}$$

By application of interface energy balance at x=L leads to:

$$(\frac{dT}{dx})_{x=L} = 0 \tag{6}$$

The solution to 4 is split into a homogeneous and particular solution:

$$T = T_h + T_p \tag{7}$$

By substitution of 7 in 4, the expression modifies as:

$$\frac{d^2T_h}{dx^2} - \frac{\bar{h}_{rad}}{kth}T_h + \frac{d^2T_p}{dx^2} - \frac{\bar{h}_{rad}}{kth}T_p = \frac{-\dot{q}^{"}}{kth} - \frac{\bar{h}_{rad}}{kth}T_\infty \tag{8}$$

The solution to homogeneous differential equation can be expressed as:

$$\frac{d^2T_h}{dx^2} - \frac{\bar{h}_{rad}}{kth}T_h = 0 \tag{9}$$

which is an exponential:

$$T_h = Cexp(mx) \tag{10}$$

Substitution of 10 in 9 leads to:

$$Cm^2exp(mx) - \frac{\bar{h}_{rad}}{kth}Cexp(mx) = 0$$

(11)

which can be evaluated as:

$$m^2 = \frac{\bar{h}_{rad}}{kth} \tag{12}$$

There are two solutions corresponding to 12 as:

$$T_h = C_1exp(mx) + C_2exp(-mx) \tag{13}$$

where C_1 and C_2 are undetermined coefficients and m can be expressed as:

$$m = \sqrt{\frac{\bar{h}_{rad}}{kth}} \tag{14}$$

The solution to particular differential equation is given as:

$$\frac{d^2T_p}{dx^2} - \frac{\bar{h}_{rad}}{kth}T_p = \frac{-\dot{q}^{"}}{kth} - \frac{\bar{h}_{rad}}{kth}T_\infty \tag{15}$$

is by inspection as:

$$T_p = \frac{\dot{q}^{"}}{\bar{h}_{rad}} + T_\infty \tag{16}$$

By substitution of 13 and 16 in 7, the expression can be evaluated as:

$$T = C_1exp(mx) + C_2exp(-mx) + \frac{\dot{q}^{"}}{\bar{h}_{rad}} + T_\infty \tag{17}$$

Substituting 17 in 6, leads to:

$$C_1mexp(mL) - C_2mexp(-mL) = 0 \tag{18}$$

```
    $UnitSystem SI MASS DEG PA C J
$Tabstops 0.2 0.4 0.6 0.8 3.5
"Input Parameters"
q_flux=900 [W/m^2] "solar flux"
th=1 [mm]*convert(mm,m) "thickness of plate"
L=2 [cm]*convert(cm,m) "half-length of plate"
k=10 [W/m-K] "conductivity of plate"
Rc=1e-4 [K-m^2/W] "contact resistance at clamped edges"
T_c=converttemp(C,K,20[C]) "casing temperature"
T_infinity=converttemp(C,K,20[C]) "surrounding temperature"
e=1 [-] "emissivity"
T_bar=300 [K] "average temperature"
h_rad=4*sigma#*e*T_bar^3 "radiation heat transfer coefficient"
"Defination of Boundary Conditions"
m=sqrt(h_rad/(k*th)) "Fin Constant"
(T_c-C_1-C_2-q_flux/h_rad-T_infinity)/Rc=-k*(C_1*m-C_2*m) "Boundary Condition for x=0"
C_1*m*exp(m*L)-C_2*m*exp(-m*L)=0 "Boundary Conditions for x=L"
"solution"
{x_bar=0 [-]} "dimensionless position"
x=x_bar*L "position"
T=C_1*exp(m*x)+C_2*exp(-m*x)+q_flux/h_rad+T_infinity "temperature solution"
T_Celsius=converttemp(K,C,T) "in C"
```

Listing 1: EES Code for analysis of Flux Meter

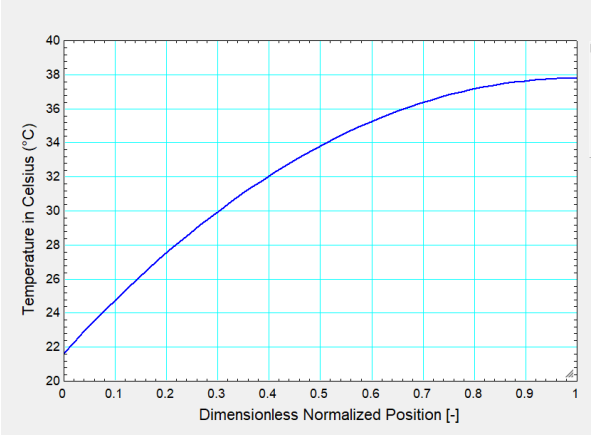


Figure 3: Dimensionless Normal Position vs. Temperature Graph

The following figure illustrates the position vs. temperature graph

```
    restart;
ODE:=diff(diff(T(x),x)-h_rad*T(x)/(k*th)=-q_flux/(k*th)-h_rad*T_infinity/(k*th);
Ts:=dsolve(ODE);

BC1:=(T_c-rhs(eval(Ts,x=0)))/Rc=-k*rhs(eval(diff(Ts,x),x=L));

BC2:=rhs(eval(diff(Ts,x),x=L))=0;
```

Listing 2: Maple Code for Analysis of Flux Meter

The Maple solution (not shown here) is embedded back in EES code to evaluate the curve for which the code can be expressed as:

```
    $UnitSystem SI MASS DEG PA C J
$Tabstops 0.2 0.4 0.6 0.8 3.5
"Input Parameters"
q_flux=900 [W/m^2] "solar flux"
th=1 [mm]*convert(mm,m) "thickness of plate"
L=2 [cm]*convert(cm,m) "half-length of plate"
k=10 [W/m-K] "conductivity of plate"
Rc=1e-4 [K-m^2/W] "contact resistance at clamped edges"
T_c=converttemp(C,K,20[C]) "casing temperature"
T_infinity=converttemp(C,K,20[C]) "surrounding temperature"
e=1 [-] "emissivity"
T_bar=300 [K] "average temperature"
h_rad=4*sigma#*e*T_bar^3 "radiation heat transfer coefficient"
"Defination of Boundary Conditions"
m=sqrt(h_rad/(k*th)) "Fin Constant"
(T_c-C_1-C_2-q_flux/h_rad-T_infinity)/Rc=-k*(C_1*m-C_2*m) "Boundary Condition for x=0"
C_1*m*exp(m*L)-C_2*m*exp(-m*L)=0 "Boundary Conditions for x=L"
"solution"
{x_bar=0 [-]} "dimensionless position"
x=x_bar*L "position"
T=C_1*exp(m*x)+C_2*exp(-m*x)+q_flux/h_rad+T_infinity "temperature solution"
T_Celsius=converttemp(K,C,T) "in C"

(T_c-C_2-C_1-1/h_rad*(q_flux+h_rad*T_infinity))/Rc = -k*(1/k^(1/2)/th^(1/2)*h_rad
    ^(1/2)*&
exp(1/k^(1/2)/th^(1/2)*h_rad^(1/2)*L)*C_2-1/k^(1/2)/th^(1/2)*h_rad^(1/2)*exp(-1/k
    ^(1/2)/&
th^(1/2)*h_rad^(1/2)*L)*C_1) "boundary condition at x=0"
1/k^(1/2)/th^(1/2)*h_rad^(1/2)*exp(1/k^(1/2)/th^(1/2)*h_rad^(1/2)*L)*C_2-1/k^(1/2)/th
    ^(1/2)*&
h_rad^(1/2)*exp(-1/k^(1/2)/th^(1/2)*h_rad^(1/2)*L)*C_1 = 0 "boundary condition at x=L"
T = exp(1/k^(1/2)/th^(1/2)*h_rad^(1/2)*x)*C_2+exp(-1/k^(1/2)/th^(1/2)*h_rad^(1/2)*x)*
    C_1&
+1/h_rad*(q_flux+h_rad*T_infinity) "solution"
x=x_bar*L "position"
```

```
32 T_Celsius=converttemp(K,C,T) "in C"
```

Listing 3: EES code with Maple Solution Embedded

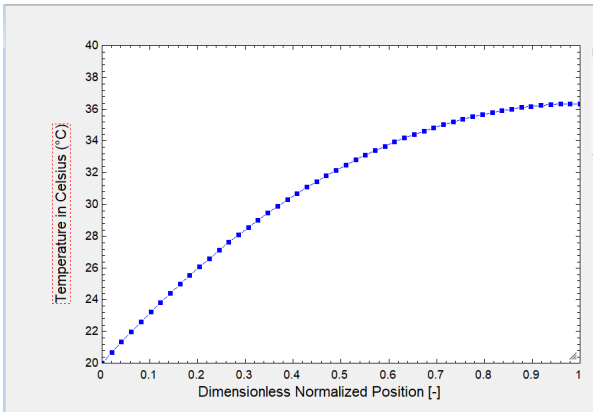


Figure 4: Dimensionless Normal Position vs. Temperature Graph using Maple Solution

The calibration curve is devised between the heat flux and temperature solution which results in a linear curve as: The unceratinity in the measurement is approximately:

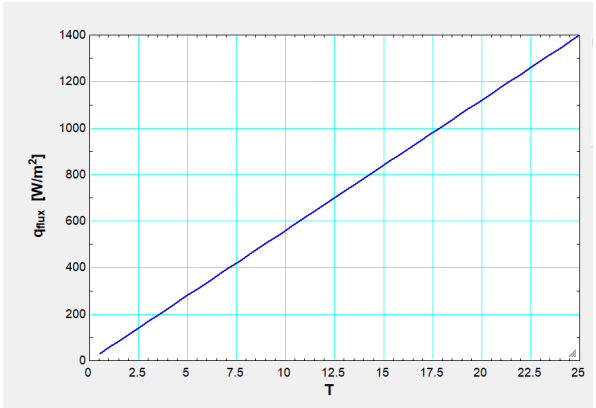


Figure 5: Heat Flux and Temperature Solution Curve

$$\delta \dot{q}^{"} = \frac{\partial \dot{q}^{"}}{\delta \Delta T} \delta \Delta T \tag{19}$$

Where $\frac{\partial \dot{q}^{"}}{\delta \Delta T} = 55 W/m^2 K$ as from 5. Therefore, the uncertainty in the heat flux is approximately 22 $\mathrm{W/m^2}$.

Conclusion

This study successfully analyzed the heat transfer characteristics of a flux meter through a combination of analytical derivations and numerical simulations. The governing differential equation was rigorously derived, accounting for radiative heat loss and boundary conditions, and its solution provided a clear understanding of the temperature distribution within the plate. The analytical results were validated using EES and Maple, ensuring accuracy and reliability. The calibration curve established a linear relationship between heat flux and temperature difference, serving as a practical tool for real-world applications. Furthermore, the uncertainty analysis underscored the sensitivity of heat flux measurements to temperature variations, emphasizing the need for precise instrumentation. The findings of this study contribute to the field of thermal measurement by offering a robust framework for flux meter design and operation. The integration of analytical and numerical methods not only enhances

the theoretical understanding of heat transfer in such systems but also provides a scalable approach for similar engineering problems. Future work could explore the effects of varying emissivity or plate geometry to further optimize the flux meter's performance.

References

[1] Nellis, G., and Klein, S. *Heat Transfer*, 1st ed. Cambridge University Press, 2009.