Similarity Solution for Momentum Transfer over Falkner-Skan Wedge Flow

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Abstract

This document provides an extensive analysis of the similarity transformation approach for solving laminar boundary layer flow over a semi-infinite vertical flat plate with constant wall temperature. By introducing a similarity variable, the governing partial differential equations (PDEs) are reduced to the Falkner-Skan equation—a third-order nonlinear ordinary differential equation (ODE). The numerical solution, obtained using Maple 2022, reveals critical boundary layer dynamics, including velocity profiles, displacement/momentum thicknesses, and wall shear stress. Validated against experimental data, the results demonstrate strong agreement, reinforcing the efficacy of similarity methods in simplifying complex transport phenomena. Key parameters such as local skin friction coefficients (c_f) and Reynolds number (Re_x) dependencies are derived, offering practical insights for engineering applications involving wedge flows, stagnation points, and bluff-body aerodynamics. This work bridges theoretical fluid mechanics with computational precision, providing a robust framework for viscous flow analysis under forced convection.

The idea of similarity transformation is to transform a complex partial differential equation to an ordinary differential equations by combining the variables in a special manner. The transformation by similarity variable is typically successful provided the similarity variable remains an independent variable. Through transformation, the number of independent variables are reduced by one, and hence for said reason, two of the original boundary conditions reduce to one.

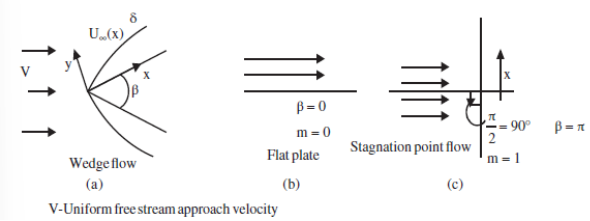


Figure 1: Wedge flow with various wedge angles Onur (2023)

Consider 1, where symmetrical fluid flow is observed. The flow is assumed to be laminar, steady, and incompressible. The similarity solution for momentum transfer over a wedge is a classical problem introduced by Falkner and Skan (1931). The boundary layer equations shall have similarity solutions provided at least one dimension extends to an infinity. The wedge geometry follows the said characteristic. The fluid velocity and pressure upstream of wedge are considered uniform. Yet, the fluid flow outside the boundary layer is inviscid. and the velocity and pressure about the velocity boundary layer vary horizontally along the wedge.

The continuity and momentum equations corresponding to said configuration are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

The momentum equation in horizontal direction is given as:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{dp_\infty}{dx} \tag{2}$$

The pressure gradient is preserved in the momentum equation since external flow is accelerated over the wedge. The pressure distribution of potential flow outside the velocity layer is typically governed by the Bernoulli's equation given as

$$\frac{1}{\rho}\frac{dp}{dx} = -U_\infty \frac{dU_\infty}{dx} \tag{3}$$

Solution is required for the freestream velocity U_∞ for inviscid flow region. Using potential flow theory, the solution to inviscid flow outside the thin boundary layer can be obtained. The potential flow theory is extensively discussed in Evans (1968), where the expression for the freestream velocity U_∞ at the edge of boundary layer is

$$U_\infty = Cx^m \tag{4}$$

C is the constant, and the exponent m is related to the wedge angle β as:

$$m = \frac{\beta/\pi}{2-\beta/\pi} = \frac{\beta}{2\pi-\beta} = \frac{x}{U_\infty}\frac{dU_\infty}{dx} \tag{5}$$

The angle β is given in radians. The angle β can be expressed in terms of parameter m as

$$\beta = \frac{2\pi m}{m+1} \tag{6}$$

The freestream velocity U_∞ varies with the distance from the wedge tip.

1. For $\beta > 0$ (measured in radians), the freestream velocity increases along the wedge surface
2. For $\beta < 0$ (measured in radians), the freestream velocity decreases along the wedge surface. For a wedge of negative opening angle is fairly impossible to exist, however the situation can be realized for suction being experienced by the boundary layer as implied in
3. When For $\beta = 0$ (measured in radians), the flow is parallel to the plate as implied in 1(b)
4. For $\beta = \pi$ (measured in radians), the flow is considered normal to the flat plate, where m=1, corresponding to flow over flat plate being set normal to the flow direction of freestream velocity. This is consistent with the flow close to the stagnation point of some bluff body as implied in 1(c)

At the edge of the boundary layer, the freestream velocity distribution is given as

$$\frac{dp_\infty}{dx} = -\rho U_\infty \frac{dU_\infty}{dx} = -\rho C x^m (Cmx^{m-1}) = -\frac{\rho m}{x}U_\infty^2 \tag{7}$$

Combining the momentum equation with 7, the following form of momentum boundary layer equation can be obtained

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \frac{mU_\infty^2}{x} \tag{8}$$

The boundary conditions for momentum equation are as follows:

$$\begin{aligned} u(x,0) &= 0 \\ v(x,0) &= 0 \\ u(x,\infty) &= U_\infty(x) = Cx^m \end{aligned}$$

discovered the similarity transformation for the wedge flow and presented the numerical solution. The similarity variable η can be defined as

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} = y\sqrt{\frac{C}{\nu}x^{m-1}} = y\sqrt{\frac{C}{\nu}}x^{m-1/2} \tag{9}$$

and dimensionless stream function Ψ which satisfies the continuity equation cen be defined as

$$\Psi = \sqrt{\nu x U_\infty f(\eta)} = \sqrt{\nu C x^{m+1}} f(\eta) \tag{10}$$

The x-component of velocity u is given as

$$u = \frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial \eta}\left(\sqrt{\nu C}x^{m+1/2}f(\eta)\right)\frac{\partial}{\partial y}\left(y\sqrt{\frac{C}{\nu}}x^{m-1/2}\right)$$

$$u = \left(\sqrt{\nu C}x^{m+1/2}\frac{df}{d\eta}\right)\left(\sqrt{\frac{C}{\nu}}x^{m-1/2}\right) = Cx^m\frac{df}{d\eta} = U_\infty\frac{df}{d\eta} \tag{11}$$

The first derivative of u with respect to y is

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial \eta}\left(\sqrt{\nu C}x^m\frac{df}{d\eta}\right)\frac{\partial}{\partial y}\left(y\sqrt{\frac{C}{\nu}}x^{m-1/2}\right)$$

$$\frac{\partial u}{\partial y} = Cx^m\frac{d^2f}{d\eta^2}\sqrt{\frac{Cx^m}{\nu x}} = U_\infty\sqrt{\frac{U_\infty}{\nu x}}\frac{d^2f}{d\eta^2} \tag{12}$$

The second derivative of u with respect tp y is

$$\nu\frac{\partial^2 u}{\partial y^2} = \nu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \nu\frac{\partial}{\partial y}\left[U_\infty\sqrt{\frac{U_\infty}{\nu x}}\frac{d^2f}{d\eta^2}\right] = \nu\frac{d}{d\eta}\left[U_\infty\sqrt{\frac{U_\infty}{\nu x}}\frac{d^2f}{d\eta^2}\right]\frac{\partial \eta}{\partial y}$$

$$\nu\frac{\partial^2 u}{\partial y^2} = \nu\frac{d}{d\eta}\left[U_\infty\sqrt{\frac{U_\infty}{\nu x}}\frac{d^2f}{d\eta^2}\right]\left[\sqrt{\frac{U_\infty}{\nu x}}\right]$$

$$\nu\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{x}\frac{df^3}{d\eta^3} \tag{13}$$

The first partial derivative of u with respect to x is

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(Cx^m\frac{df}{d\eta}\right) = \frac{\partial}{\partial x}\left(Cx^m\frac{df}{d\eta} + (Cx^m)\frac{\partial}{\partial x}\left(\frac{df}{d\eta}\right)\right)$$

$$= Cmx^{m-1}\frac{df}{d\eta} + (Cx^m)\frac{d^2f}{d\eta^2}\frac{\partial \eta}{\partial x}$$

$$= Cmx^{m-1}\frac{df}{d\eta} + (Cx^m)\frac{d^2f}{d\eta^2}\frac{\partial}{\partial x}\left[y\sqrt{\frac{Cx^m}{vx}}\right]$$

$$= Cmx^{m-1}\frac{df}{d\eta} + (Cx^m)\frac{d^2f}{d\eta^2}\left[\frac{y}{2}(m-1)\sqrt{\frac{C}{v}}x^{m-3}\right]$$

$$= m\left(\frac{Cx^m}{x}\frac{df}{d\eta} + (Cx^m)\frac{d^2f}{d\eta^2}\left[\frac{m-1}{2}y\sqrt{\frac{Cx^{m-1}}{vx^2}}\right]\right)$$

$$= m\left(\frac{Cx^m}{x}\frac{df}{d\eta} + (Cx^m)\frac{d^2f}{d\eta^2}\left[\frac{m-1}{2}\sqrt{\frac{Cx^{m-3}}{v}}\right]\right)$$

$$= m\left(\frac{Cx^m}{x}\frac{df}{d\eta} + (Cx^m)\frac{d^2f}{d\eta^2}\left[\frac{m-1}{2}y\sqrt{\frac{Cx^{m-1}}{v}}x^{-2}\right]\right)$$

$$\frac{\partial u}{\partial x} = m\frac{df}{d\eta}\frac{U_\infty}{x} + \frac{U_\infty}{x}\frac{d^2f}{d\eta^2}\eta\left(\frac{m-1}{2}\right). \tag{14}$$

The y-component of velocity v is obtained as follows:

$$v = -\frac{\partial \Psi}{\partial x} = -\frac{\partial}{\partial x}\left[\sqrt{vC}x^{\frac{m+1}{2}}f(\eta)\right]$$

$$= -\left[\frac{\partial}{\partial x}\left(\sqrt{vC}x^{\frac{m+1}{2}}f(\eta) + \sqrt{vC}x^{\frac{m+1}{2}}\frac{df}{d\eta}\frac{\partial \eta}{\partial x}\right)\right]$$

$$= -\left[\frac{\partial}{\partial x}\left(\sqrt{vC}x^{\frac{m+1}{2}}f(\eta) + \sqrt{vC}x^{\frac{m+1}{2}}\frac{df}{d\eta}\frac{\partial}{\partial x}\left(y\sqrt{\frac{C}{v}}x^{\frac{m-1}{2}}\right)\right)\right]$$

$$= -\left\{\sqrt{vC}\left(\frac{m+1}{2}x^{\frac{m-1}{2}}f(\eta)+\sqrt{vC}x^{\frac{m+1}{2}}\frac{df}{d\eta}\left[y\sqrt{\frac{C}{v}}\left(\frac{m-1}{2}\right)x^{\frac{m-3}{2}}\right]\right\}\right.$$

$$= -\left\{\sqrt{\frac{vCx^m}{x}}\left(\frac{m+1}{2}f(\eta)+\sqrt{vC}\frac{x^m v_x}{vx^2}\frac{df}{d\eta}\left[\frac{m-1}{2}y\sqrt{\frac{Cx^{m-1}}{vx^2}}\right]\right\}\right.$$

$$= -\left\{\sqrt{\frac{vCx^m}{x}}\left(\frac{m+1}{2}f(\eta)+\sqrt{vC}\frac{x^m v_x}{x}\frac{df}{d\eta}\left[\frac{m-1}{2}\eta\right]\right\}\right.$$

$$= -\left\{\sqrt{\frac{vU_\infty x^{m+1}}{x}}\left(\frac{m+1}{2}f(\eta)+\sqrt{\frac{U_\infty v}{x}}\frac{df}{d\eta}\left[\frac{m-1}{2}\eta\right]\right\}\right.$$

$$v = -\sqrt{\frac{U_\infty v}{x}}\left(\frac{m+1}{2}\right)\left[f-\frac{1-m}{1+m}\eta\frac{df}{d\eta}\right]. \tag{15}$$

The pressure gradient with respect to x is

$$\frac{1}{\rho}\frac{dp}{dx} = -U_\infty\frac{dU_\infty}{dx} = -mC^2x^{2(m-1)} \tag{16}$$

Where $U_\infty = Cx^m$ and substituting the above equations into the momentum boundary layer equation, the constant C can be evaluated and cancellation of x variable, the Falkner-Skan nonlinear ordinary differential equation can be obtained in terms of $f(\eta)$

$$\frac{d^3f}{d\eta^3}+\frac{1}{2}(m+1)f\frac{d^2f}{d\eta^2}+m\left[1-\left(\frac{df}{d\eta}\right)^2\right]=0 \tag{17}$$

The boundary conditions are

$$f(0)=0$$
$$\frac{df(0)}{d\eta}=0$$
$$\frac{df(\infty)}{d\eta}=1$$

This is a nonlinear boundary value problem with the problem solved by Hartree (1937) . The missing $f^{"}(0)$ is approximated by Forbrich (1982) as

$$f^{"}(0) = \frac{1}{\sqrt{2-\frac{\beta}{\pi}}}\left[\frac{\frac{\beta}{\pi}+0.1988377}{0.8160218}\right]^{0.53394399} \tag{18}$$

```
> restart;
with(plots):

# Data from Table 7.2 (excluding infinite values)
beta_values := [-0.625, -0.314, 0, evalf(Pi/5), evalf(Pi/2), evalf(Pi), evalf(5*Pi/3)
    ]:
m_values := [-0.0904, -0.0476, 0, 0.111, 0.333, 1, 5]:
fpp_Kays := [0.0, 0.2203, 0.3321, 0.51182, 0.75713, 1.2325, 2.6852]:
fpp_Maple := [0.0, 0.2203, 0.3320, 0.5116, 0.7574, 1.2325, 2.6852]:

# Create data points
data_Kays := [seq([beta_values[i], fpp_Kays[i]], i=1..7)]:
data_Maple := [seq([beta_values[i], fpp_Maple[i]], i=1..7)]:

# Empirical formula with protection against singularities
```

```
fpp_formula := proc(beta)
    local x;
    x := beta/evalf(Pi);
    if x >= 2 then return undefined end if; # Avoid division by zero
    1/sqrt(2 - x) * ((x + 0.1988377)/0.8160218)^0.53394399;
end proc:

# Create plots
p1 := pointplot(data_Kays, symbol=solidcircle, symbolsize=15, color=blue, legend="Kays
     et al."):
p2 := pointplot(data_Maple, symbol=solidbox, symbolsize=15, color=red, legend="Maple
    2020"):
p3 := plot(fpp_formula, -0.7..evalf(5*Pi/3)+0.1, color=green, thickness=2, legend="
    Empirical Formula",
          discont=true, adaptive=true):

# Display combined plot
display([p1, p2, p3],
       title="Falkner-Skan Solutions: f''(0) vs ",
       labels=[" ", "f''(0)"],
       labeldirections=[horizontal, vertical],
       axes=boxed,
       view=[-0.7..6.5, 0..3]);
```

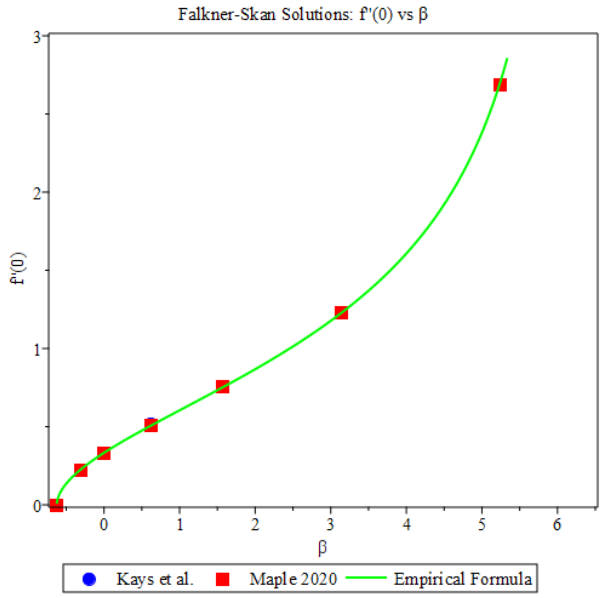


Figure 2: Graphical Solution for Falkner-Skan Equation

Table 1: Comparison of $\frac{d^2f(0)}{d\eta^2}$ values from Kays et al. [26] and Maple 2020

β	m	$\frac{d^2f(0)}{d\eta^2}$		Flow Type
		Kays et al. [26]	Maple 2020	
-0.625	-0.0904	0.0	0.0	Separation
-0.314	-0.0476	0.2203	0.2203	
0	0	0.3321	0.3320	Flat plate
$\pi/5$	0.111	0.51182	0.5116	
$\pi/2$	0.333	0.75713	0.7574	
π	1	1.2325	1.2325	Stagnation flow
$5\pi/3$	5	2.6852	2.6852	
2π	∞	∞	∞	

The above Maple code can be used for solving 17. The following table presents the data validation Assuming unit depth into the paper, the local shear wall stress can be evaluated at the wedge surface as τ_w

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu\left[\left(\frac{\partial u}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)\right] = \mu\left[\left(\frac{d}{d\eta}\left(U_\infty \frac{df}{d\eta}\right)\right)\left(\sqrt{\frac{U_\infty}{\nu x}}\right)\right]_{\eta=0}$$

$$\tau_w = \frac{\mu U_\infty}{x}\sqrt{Re_x}\left(\frac{d^2f}{d\eta^2}\right)_{\eta=0} \tag{19}$$

where $Re_x = \frac{U_\infty x}{\nu}$ is the Reynolds number. The wall shear stress can be expressed conventionally in terms of dimensionless form. Local friction coefficient c_f has to be defined for said purpose

$$c_f = \frac{\tau_w}{\frac{\rho U_\infty^2}{2}} = \frac{\frac{\mu U_\infty}{x}\sqrt{Re_x}\left(\frac{d^2f}{d\eta^2}\right)_{\eta=0}}{\frac{\rho U_\infty^2}{2}} = \frac{2}{\sqrt{Re_x}}\left(\frac{d^2f}{d\eta^2}\right)_{\eta=0} \tag{20}$$

The Reynolds number is defined as:

$$Re_x = \frac{U_\infty(x)x}{\nu} = \frac{Cx^{m+1}}{\nu}. \tag{21}$$

The displacement thickness δ_1 and momentum thickness δ_2 are given by:

$$\delta_1 = \int_0^\infty \left[1 - \frac{u}{U_\infty}\right] dy \tag{22}$$

$$\delta_2 = \int_0^\infty \left(\frac{u}{U_\infty}\right)\left[1 - \frac{u}{U_\infty}\right] dy. \tag{23}$$

Using the dimensionless distance $\eta = y\sqrt{U_\infty/\nu x}$, we have:

$$dy = \sqrt{\frac{\nu x}{U_\infty}} d\eta \tag{24}$$

$$\frac{u}{U_\infty} = \frac{df}{d\eta} \tag{25}$$

The displacement and momentum thicknesses for wedge flow become:

$$\frac{\delta_1}{x} = \frac{1}{\sqrt{Re_x}}\int_0^\infty \left[1 - \frac{df}{d\eta}\right] d\eta \tag{26}$$

$$\frac{\delta_2}{x} = \frac{1}{\sqrt{Re_x}}\int_0^\infty \left(\frac{df}{d\eta}\right)\left[1 - \frac{df}{d\eta}\right] d\eta \tag{27}$$

Table 2: Solution of Falkner-Skan Equation 17 using m=1

η	t	$\frac{df}{d\eta}$	$\frac{d^2t}{d\eta^2}$
0	0	0	1.2330
0.1000	0.0060	0.1183	1.1332
0.2000	0.0233	0.2267	1.0349
0.3000	0.0510	0.3254	0.9390
0.4000	0.0881	0.4146	0.8468
0.5000	0.1336	0.4949	0.7587
0.6000	0.1867	0.5666	0.6755
0.7000	0.2466	0.6302	0.5977
0.8000	0.3125	0.6863	0.5256
0.9000	0.3837	0.7355	0.4593
1.0000	0.4594	0.7783	0.3986
1.1000	0.5392	0.8154	0.3437
1.2000	0.6223	0.8473	0.2943
1.3000	0.7085	0.8745	0.2506
1.4000	0.7971	0.8975	0.2119
1.5000	0.8879	0.9170	0.1779
1.6000	0.9804	0.9333	0.1483
1.7000	1.0744	0.9468	0.1228
1.8000	1.1697	0.9580	0.1011
1.9000	1.2660	0.9671	0.0827
2.0000	1.3631	0.9746	0.0672
2.1000	1.4608	0.9806	0.0542
2.2000	1.5591	0.9855	0.0435
2.3000	1.6579	0.9894	0.0348
2.4000	1.7570	0.9925	0.0277
2.5000	1.8564	0.9950	0.0220
2.6000	1.9560	0.9969	0.0175
2.7000	2.0558	0.9985	0.0139
2.8000	2.1557	0.9997	0.0111
2.9000	2.2557	1.0007	0.0090
3.0000	2.3558	1.0016	0.0073
3.1000	2.4560	1.0022	0.0061
3.2000	2.5563	1.0028	0.0052
3.3000	2.6566	1.0033	0.0045
3.4000	2.7569	1.0037	0.0041
3.5000	2.8573	1.0041	0.0038
3.6000	2.9577	1.0045	0.0036
3.7000	3.0582	1.0048	0.0034
3.8000	3.1587	1.0051	0.0034
3.9000	3.2592	1.0055	0.0034
4.0000	3.3598	1.0058	0.0034
4.1000	3.4604	1.0062	0.0034
4.2000	3.5610	1.0065	0.0035
4.3000	3.6617	1.0068	0.0035
4.4000	3.7624	1.0072	0.0036

4 can be applied near the leading edge of a blunt object such as a cylinder or a sphere. The freestream velocity on stagnation point of cylinder and sphere can be predicted using potential flow theory Schlichting and Gersten (2017).

$$U_\infty = 2Vsin\left(\frac{x}{R}\right) \approx 4V\left(\frac{x}{d}\right) \quad \text{for cylinder}$$

$$U_\infty = \frac{3}{2}Vsin\left(\frac{x}{R}\right) \approx 3V\left(\frac{x}{d}\right) \quad \text{for sphere}$$

where V is the oncoming normal velocity, x is the distance along the surface measured from the stagnation

point, and D is the cylindrical diameter. U_∞ is the velocity outside the boundary layer.

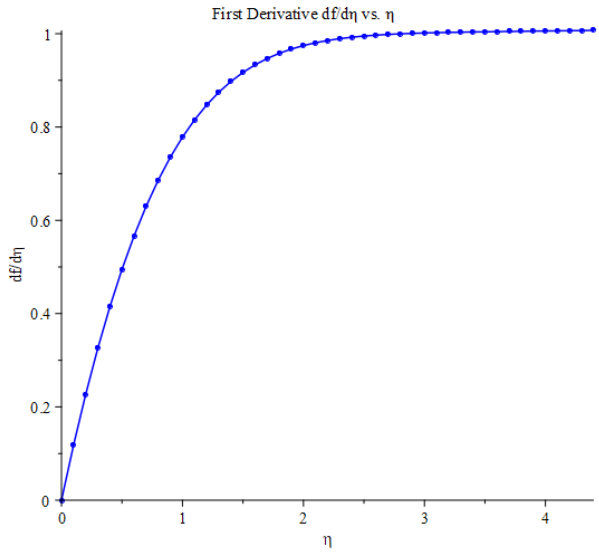


Figure 3: Plot for η and $\frac{df}{d\eta}$

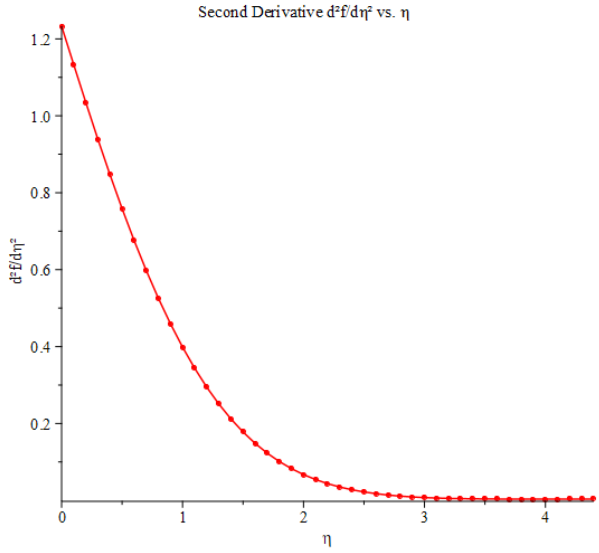


Figure 4: Plot for η and $\frac{d^2f}{d\eta^2}$

Conclusion

The application of similarity transformation has successfully simplified the analysis of momentum boundary layer flow over an infinite wedge surface, reducing the governing PDEs to a manageable ODE. The numerical solution of the resulting Falkner-Skan equation accurately predicts the momentum boundary later thickness and the velocity distribution within the boundary layer and agrees well with experimental data. Key boundary layer properties, including boundary layer thickness, wall shear stress, and local skin friction were computed and expressed as functions of the local Reynolds number. The equations can be utilized for predicting he freestream velocities about bliunt bodies using potential flow theory

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