One-Dimensional Solutions in Convective Heat Transfer: Heat Transfer in Couette Flow

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Abstract

This study investigates the coupled convective heat transfer and viscous dissipation in Couette flow configuration between two infinite parallel plates. The upper plate moves with constant velocity, while the lower plate remains stationary and is maintained at a higher temperature. A one-dimensional analytical solution is derived for velocity and temperature distributions under laminar, steady, and incompressible flow conditions with constant fluid properties. The governing equations, including the momentum and energy equations, are simplified using boundary-layer assumptions and solved symbolically in MATLAB. Dimensionless expressions for temperature profile, heat flux, Nusselt number, and Fanning friction factor are obtained as functions of the Brinkman and Eckert numbers. The results provide insights into the influence of viscous dissipation and Brinkman number on heat transport phenomena, with implications for design and analysis of microfluidic devices and thermal systems where viscous heating is non-negligible.

Keywords: Couette flow; viscous dissipation; Brinkman number; heat transfer; Nusselt number; MATLAB modeling

Couette flow is the flow of some viscous fluid between two surfaces in space, where one of the surfaces moves tangentially relative to the other. Due to relative motion of surface, shear stress is induced in the surfaces which induces fluid flow. The terminology of Couette flow can be further enhanced for applied pressure gradients in the flow direction. The phenomenon is extensively applied in viscometry and for demonstrating the approximations of reversibility Guyon et al. (2001); Heller (1960). The Couette flow was first considered by Maurice Couette. Couette flow models provide simple heat transfer model for flow between two parallel plates White and Majdalani (2021). The physical importance of the model improves by analyzing over wide variety of geometries, especially in understanding parameters governing velocity and temperature distribution. Consider 1, consisting of two infinite

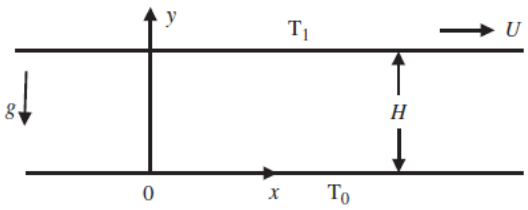


Figure 1: Heat Transfer in Couette Flow Onur (2023)

parallel impermeable plates. An incompressible, constant property fluids fill the space between the plates. The lower plate is kept stationary, while the upper plate moves with uniform velocity U along positive x-direction. The temperature of lower plate is greater than that of upper plate. Considering the viscous dissipation into account, the velocity distribution, temperature distribution, heat flux at both surfaces, location of maximum temperature, Nusselt number and Fanning friction factor can be evaluated for lower surface while neglecting the influence of gravity or pressure variation in fluid.

As stated above, the plates are of infinite length and there occurs no variation in x- or z- direction and there is no motion in z-direction as well, thus $w = 0$. The viscous dissipation in flow leads to generation of energy and the fluid flow tends to fully develop since all fluid particles move parallel to plates. The velocity distribution,

once determined ultiamtely leads to determination of temperature distribution, heat flux and Nusselt number. For effective analysis of phenomenon, following assumptions may be taken into account:

- Flow remains steady throughout
- The flow rests in laminar regime
- The fluidic properties such as density ρ, heat capacity c_p, conductivity k and viscosity (dynamic, μ) remain constant throughout.
- There are no pressure gradients in flow direction
- Since plates are infinite, so the edge effects are ignored $\frac{\partial}{\partial z} = 0$
- Since plates are infinite, so the end effects are ignored $\frac{\partial}{\partial x} = 0$
- The fluid around plates is Newtonian
- The flow is fully developed with the velocity and temperature distribution not changing along flow direction, which implies $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial T}{\partial x} = 0$
- No body forces exist in the x-direction
- There is no motion in z-direction, $w = 0$

The continuity equation can be expressed as

$$\frac{D\rho}{Dt} + \rho(\vec{\nabla}.\vec{V}) = 0 \tag{1}$$

or

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \tag{2}$$

For an incompressible fluid

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial y} = \frac{\partial \rho}{\partial z} = 0$$

The final continuity equation becomes

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

Since the two parallel plates are of infinite length, the edge and end effects are not taken into consideration, and since there is no motion in z-direction as $w = 0$, so

$$\frac{\partial w}{\partial z} = 0 \qquad \text{no edge effects}$$
$$\frac{\partial u}{\partial x} = 0 \qquad \text{No end effects and hydrodynamically fully developed fluid flow}$$

From the above simplifications, the continuity equation reduces to following format:

$$\frac{\partial v}{\partial y} = 0$$

On the basis of such simplifications, the continuity equation reduces to

$$v(x, y) = f(x)$$

By the no-slip boundary condition, since it is applied at the walls

$$y = 0 \qquad v = 0 \qquad \text{Impermeable wall}$$

which implies $f(x) = 0$, so the y-component of velocity becomes

$$v(x, y) = 0$$

which further implies that the streamlines are parallel in flow field, and the y-component of velocity is zero elsewhere. Considering the y-component in the momentum equation

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho f_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

The y-component of velocity reduces to

$$0 = \rho f_y - \rho g y$$

Considering the x-component of momentum equation

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$
\begin{aligned}
f_x &= 0 \\
\frac{\partial u}{\partial t} &= 0 \qquad \text{Steady Flow} \\
\frac{\partial u}{\partial x} &= 0 \qquad \text{Hydrodynamically fully developed flow} \\
\frac{\partial p}{\partial x} &= 0 \qquad \text{No variation in axial pressure in flow} \\
\frac{\partial^2 u}{\partial x^2} &= 0 \qquad \text{since } \frac{\partial u}{\partial x} = 0 \\
\frac{\partial^2 u}{\partial z^2} &= 0 \qquad \text{since } \frac{\partial u}{\partial z} = 0
\end{aligned}
$$

The horizontal component of momentum equation reduces to

$$\frac{\partial^2 u}{\partial y^2} = 0$$

The corresponding boundary conditions are:

$$
\begin{aligned}
u(0) &= 0 \\
u(H) &= U
\end{aligned}
$$

Once plugged in MATLAB solver using the following code, the resultant expression becomes:

```
clc;
clear all;
syms u(y) y h U k T(y) mu F T0 T1

de=diff(u,y,2)==0;

BC(1)=u(0)==0;
BC(2)=u(H)==U;

BC=[BC(1),BC(2)]

u=dsolve(de,BC)
```

Listing 1: MATLAB code for evaluating the velocity profile

$$\frac{u(y)}{U} = \frac{y}{H}$$

From the velocity distribution once known, the energy equation coupled with viscous dissipation can be taken into consideration:

$$\rho c_p \left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right] = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + q''' + \mu\Phi$$

where

$$\Phi = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2$$

$$
\begin{aligned}
q''' &= 0 \qquad \text{No energy generation} \\
\frac{\partial T}{\partial t} &= 0 \qquad \text{Steady Flow} \\
\frac{\partial T}{\partial x} &= 0 \qquad \text{No end effects} \\
\frac{\partial T}{\partial z} &= 0 \qquad \text{No edge effect and w=0 since there is no motion in z-direction} \\
v\frac{\partial T}{\partial y} &= 0 \qquad \text{Since v=0 due to continuity equation} \\
\frac{\partial^2 T}{\partial x^2} &= 0 \qquad \text{Negligible axial conduction} \\
\frac{\partial^2 T}{\partial z^2} &= 0 \qquad \text{No edge effects. Since there are no edge effects so } \frac{\partial T}{\partial z} = 0 \text{and since} \frac{\partial T}{\partial z} = 0, \frac{\partial^2 T}{\partial z^2} = 0
\end{aligned}
$$

From all the following aspects, the energy equation reduces to the following expression

$$k\frac{d^2T}{dy^2} + \mu\Phi = 0$$

Considering the assumptions made, the viscous dissipation term reduces to

$$\Phi = \left(\frac{\partial u}{\partial y}\right)^2$$

The energy equation takes the following shape

$$\frac{d^2T}{dy^2} = -\left(\frac{\mu}{k}\right)\left(\frac{U}{H}\right)^2$$

The corresponding boundary conditions are

$$T(0) = T_o$$
$$T(H) = T_1$$

```
clc;
clear all;
syms u(y) y h U k T(y) mu F T0 T1

F=diff(u,y)^2;
de=k*diff(T,y,2)+(mu)*F==0;

BC(1)=T(0)=T0;
BC(2)=T(H)=T1;

BC=[BC1,BC2];

sol=dsolve(de,BC);
sol=expand(sol);
```

Listing 2: MATLAB code for evaluating temperature profile

On running the above MATLAB following equation is yielded

$$T_0 + \frac{y\left(\mu U^2 - 2T_0 k + 2T_1 k\right)}{2Hk} - \frac{U^2 \mu y^2}{2H^2 k}$$

On further simplification

$$T_0 - \frac{T_0 y}{H} + \frac{T_1 y}{H} + \frac{U^2 \mu y}{2Hk} - \frac{U^2 \mu y^2}{2H^2 k}$$

The final solution can be expressed as

$$T - T_o = \left(\frac{y}{H}\right)[T_1 - T_o] + \left(\frac{\mu U^2}{2k}\right)\left(\frac{y}{H}\right)\left[1 - \left(\frac{y}{H}\right)\right]$$

The first term arises from conduction, the second term arises from viscous dissipation in te fluid flow. IN dimensionless form, the expression reduces to

$$\theta = \eta + \frac{EcPr}{2}(\eta - \eta^2)$$

where $\theta = \frac{T-T_o}{T_1-T_o}$, $\eta = \frac{y}{H}$, $Pr = \frac{\mu c_p}{k}$ and $Ec = \frac{U^2}{c_p(T_1-T_o)}$ is Eckert number. Eckert number is the measure of viscous dissipation in fluid. The product of Eckert number and Prandtl number is referred as Brinkman]n number, $Br = EcPr$. For $T_1 > T_o$, the heat flux can be evaluated at both ends of plates. For stationary (lower plate), the corresponding expression for heat flux can be written as

$$q_w^{"} = -\frac{k(T_1 - T_o)}{H}\left(1 + \frac{Br}{2}\right)$$

For the upper moving plate, the heat flux equation can be written as

$$q_w^{"} = -\frac{k(T_1 - T_o)}{H}\left(1 - \frac{Br}{2}\right)$$

Consider heat transfer on moving plate, the dimensionless temperature distribution upon examination reveals

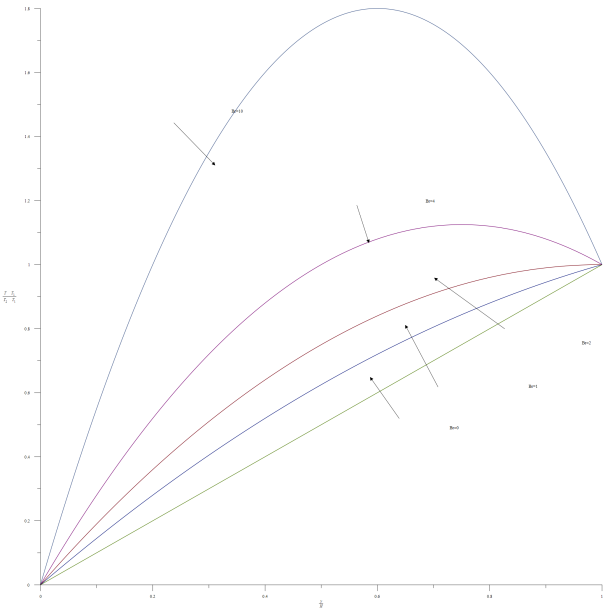


Figure 2: Dimensionless temperature distribution in Couette flow for various Brinkman numbers

heat transfer occurs from moving plate to fluid for $Br < 2$, and heat transfer is from fluid to moving plate for $Br > 2$. If $Br = 2$, no heat transfer takes place from the moving plate. No flow condition arises when $Br = 0$, where temperature distribution is a straight line and represents pure heat transfer through conduction. The location of heat transfer can be determined by the solving

$$\frac{dT}{dy} = (T_1 - T_o)\left[\frac{1}{H} + \frac{Br}{2}\left(\frac{1}{H} - \frac{2y}{H^2}\right)\right]$$

This results in

$$y_{max} = H\left[\frac{1}{2} + \frac{1}{Br}\right]$$

The second derivative can be expressed as

$$\frac{d^2T}{dy^2} = -\frac{(T_2 - T_1)}{H^2} EcPr < 0$$

and is always negative. Hence y_{max} is the location of maximum temperature, which is given as

$$\frac{T_{max} - T_o}{T_1 - T_o} = \frac{1}{2} + \frac{1}{2Br} + \frac{Br}{8} = \frac{(2 + Br)^2}{8Br}$$

Temperature increase in fluid flow is observed when Brinkman number of unity or greater is observed. For low speed fluid flow, the viscous fluids have relatively high Brinkman numbers. Except high viscous fluids, in low-speed flow, the viscous dissipation is usually neglected. The Nusselt number for lower expression can be determined provided the heat transfer coefficient $\bar{h}$ is known, which can be defined as

$$\bar{h} = \frac{q_w^{"}}{T_o - T_1} = \frac{-k\left(\frac{dT}{dy}\right)_{y=0}}{T_o - T_1} = \frac{-k(T_o - T_1)\left[\frac{1}{H} + \frac{Br}{2}\left(\frac{1}{H} - \frac{2y}{H^2}\right)\right]_{y=0}}{T_o - T_1}$$

$$\bar{h} = \frac{-k(T_o - T_1)\left[\frac{1}{H} + \frac{Br}{2}\left(\frac{1}{H}\right)\right]}{T_o - T_1}$$

$$\bar{h} = k\left[\frac{1}{H} + \frac{Br}{2}\left(\frac{1}{H}\right)\right]$$

The Nusselt number can be defined as

$$Nu_{D_H} = \frac{\bar{h}D_H}{k} = 2\left[1 + \frac{Br}{2}\right]$$

where $D_H = 2H$ is the hydraulic diameter. The heat transfer coefficient can be evaluated from the bottom plate while the Nusselt number can be associated with the moving plate

$$\bar{h} = \frac{q_w^{"}}{T_o - T_1} = \frac{-k\left(\frac{dT}{dy}\right)_{y=H}}{T_o - T_1} = \frac{-k(T_1 - T_o)\left[\frac{1}{H} + \frac{Br}{2}\left(\frac{1}{H} - \frac{2y}{H^2}\right)\right]_{y=H}}{T_o - T_1}$$

$$\bar{h} = \frac{-k(T_1 - T_o)\left[\frac{1}{H} + \frac{Br}{2}\left(\frac{1}{H}\right)\right]}{T_o - T_1}$$

$$\bar{h} = k\left[\frac{1}{H} - \frac{Br}{2}\left(\frac{1}{H}\right)\right]$$

The Nusselt number can hence be defined as

$$Nu_{D_H} = \frac{\bar{h}D_H}{k} = 2\left[1 - \frac{Br}{2}\right]$$

where $D_H = 2H$ is the hydraulic diameter. The Fanning friction factor c_f for lower plate can be determined as

$$c_f = \frac{(\tau)_{y=0}}{\frac{1}{2}\rho U^2} = \frac{\mu\left(\frac{du}{dy}\right)_{y=0}}{\frac{1}{2}\rho U^2} = \frac{\mu\frac{U}{H}}{\frac{1}{2}\rho U^2} = \frac{2\mu}{\rho UH} = \frac{4\mu}{2\rho UH} = \frac{4\mu}{\rho UD_H}$$

which reduces to following format

$$c_f = \frac{4}{Re_{D_H}}$$

where $Re_{D_H} = \frac{\rho UD_H}{\mu}$ is the Reynolds number and $D_H = 2H$ is the hydraulic number.

Conclusion

In this study, a comprehensive analytical formulation for heat transfer in Couette flow with viscous dissipation was developed. The flow field and thermal behavior between two infinite parallel plates were examined under a set of simplifying assumptions, allowing for an exact solution to the coupled momentum and energy equations. The results demonstrate that the temperature distribution in the fluid is strongly influenced by the Brinkman number, with significant deviations from linear profiles at higher viscous dissipation levels. The derived expressions for Nusselt number and Fanning friction factor provides insight on the dependence on fluid viscosity, flow velocity, and system geometry.

The analysis confirms that for Brinkman numbers greater than two, heat is transferred from the fluid to the moving plate, whereas for values less than two, the reverse occurs. A special condition arises at $Br = 2$ where no net heat transfer takes place at the upper boundary. These findings underscore the importance of accounting for viscous effects in thermal systems, especially in microscale applications and highly viscous flows. The analytical framework presented here provides a reliable benchmark for validating numerical solvers and optimizing thermal-fluid designs.

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