Heat Distribution on Circular Fin

Muhammad Abdullah Zafar Ghauri

2nd March 2025

Abstract

This study investigates the heat distribution on a circular fin using numerical methods, specifically finite difference approximation and MATLAB's pdepe solver. The governing partial differential equation incorporates thermal conduction and convective heat transfer effects. The finite difference method is implemented with first- and second-order approximations to discretize the temperature field, while boundary conditions are enforced using backward second-order schemes. The MATLAB pdepe function is also applied to solve the equation in its generalized form. Comparative results demonstrate the accuracy and efficiency of both methods in modeling transient heat distribution. Temperature profiles, mesh representations, and numerical validations are presented to highlight the thermal behavior of the fin. This work provides a structured approach to solving heat conduction problems in engineering applications.

1 Solution using Finite Difference Approximation

In the study of heat transfer, fins are surfaces that extend from an object to increase the rate of heat transfer to or from the environment by increasing convection. The amount of conduction, convection, or radiation of an object determines the amount of heat it transfers. Increasing the temperature gradient between the object and the environment, increasing the convection heat transfer coefficient, or increasing the surface area of the object increases the heat transfer. Sometimes it is not feasible or economical to change the first two options. Thus, adding a fin to an object, increases the surface area and can sometimes be an economical solution to heat transfer problems.In a circular fin, heat distribution refers to the gradual decrease in temperature from the base of the fin (where it is attached to the heat source) to its tip, with the highest heat transfer occurring near the base and progressively diminishing as the radius increases, due to the continuous heat loss to the surrounding environment through convection; this temperature gradient is described by a second-order differential equation that takes into account the fin's geometry and thermal properties. The heat distribution on a circular fin is given by the following partial differential

equation:

$$\frac{\rho c_p}{k}\frac{dT}{dt} = \frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} - \frac{2h}{kb}(T - T_u) \tag{1}$$

Subjected to intial conditions as:

$$t = 0; r = r; T = T_o \tag{2}$$

and boundary conditions as:

$$r = R_{in}; T(1) = Ts + 30t - 6t^2 \tag{3}$$

$$r = R_{out}; -k\frac{dT}{dr} = h(T - T_u) \tag{4}$$

By the application of first order and second order finite difference approximations in centered configurations:

$$\frac{dT}{dr} = \frac{T(i+1) - T(i-1)}{2\Delta r} \tag{5}$$

$$\frac{d^2T}{dr^2} = \frac{T(i+1) - 2T(i) + T(i-1)}{\Delta r^2} \tag{6}$$

Through discretization of above equations, the final equation for the heat distribution on circular fin takes shape as:

$$\frac{dT}{dr} = \frac{\frac{T(i+1)-2T(i)+T(i-1)}{\Delta r^2} + \frac{1}{r_i}(\frac{T(i+1)-T(i-1)}{2\Delta r}) - \frac{2h}{kb}(T(i) - Tu)}{\frac{\rho c_p}{k}} \tag{7}$$

By performing backward second order finite difference approximation on the boundary conditions, the expression uses the term "end" which refers to the last discretized node over the surface:

$$-k\frac{3(Tend) - 4T(end-1) + T(end-2)}{2\Delta r} = h(T(end) - Tu) \tag{8}$$

$$T(end) = \frac{\frac{2\Delta r h Tu}{k} + 4T(end-1) - T(end-2)}{(3 + \frac{2\Delta r h}{k})} \tag{9}$$

```
clc
clear
close all
%Experimental Data
kfin=0.8; %Conductivity of Fin (cal/s/cm/K)
hfin=0.0007;% Heat Transfer Coefficient of Fin (cal/s/cm^2/K)
Rin=3; %Inner Radius (cm)
Rout=9;%Outer Radius (cm)
Ts=400;%Surface Temperature (C)
Tu=30;%Inner Temperature (C)
T0=35;%Outer Temperature (C)
Nr=200;%Step Size
rhofin=0.8; %Density of Fn Material (g/cm^3)
```

Table 1: Experimental Data Values

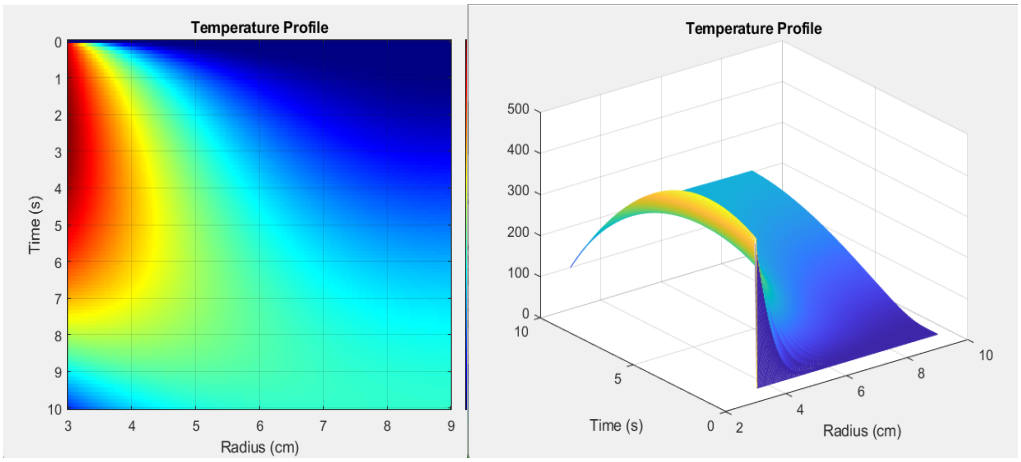
Conductivity of Fin	k_{fin}	0.8 cal/s/cm/K
Heat Transfer Coefficient of Fin	h_{fin}	0.0007 cal/s/cm^2/K
Inner Radius	R_{in}	3 cm
Outer Radius	R_{out}	9 cm
Surface Temperature	T_s	400 ° C
Inner Temperature	T_u	30 ° C
Outer Temperature	T_o	35 ° C
Step Size	Nr	200
Density of Fn Material	ρ_{fin}	0.8 g/cm^3
SpecificHeat Capacity of Fin	$c_{p,fin}$	0.5 cal/g/K

```
cpfin=0.5; %SpecificHeat Capacity of Fin (cal/g/K)
b=1;
%Defining of Step Size and Increment
r=linspace(Rin,Rout,Nr);
dr=r(2)-r(1); %delta r
t=linspace(0,10,100); %Time discretization from 1-100 s
%Simplification of Calculations
m=rhofin*cpfin/kfin;
n=2*hfin./kfin./b;
o=2*dr*hfin./kfin;
%Application of Initial Conditions
IC=T0.*ones(Nr,1);
%ODE Solver
[t,T] =ode15s(@f,t,IC,[],Rin,Rout,Ts,Tu,T0,Nr,m,n,o,dr,r);
%Recalculation
T(:,1)=Ts+30.*t-6.*t.^2;
T(:,end)=(o.*Tu+4*T(:,end-1)-T(:,end-2))./(3+o); %Second Boundary
    Condition
figure(1)
imagesc(r,t,T)
colormap jet
colorbar
grid on
title('Temperature Profile')
xlabel('Radius (cm)')
ylabel('Time (s)')
figure(2)
mesh (r,t,T)
title('Temperature Profile')
xlabel('Radius (cm)')
ylabel('Time (s)')
%Defining of function
function dTdt=f(t,T,Rin,Rout,Ts,Tu,T0,Nr,m,n,o,dr,r)
dTdt=zeros(Nr,1);
T(1)=Ts+30.*t-6.*t.^2;
T(end)=(o.*Tu+4*T(end-1)-T(end-2))./(3+o); %Second Boundary
    Condition

%Interior Conditions
for i=2:Nr-1
```

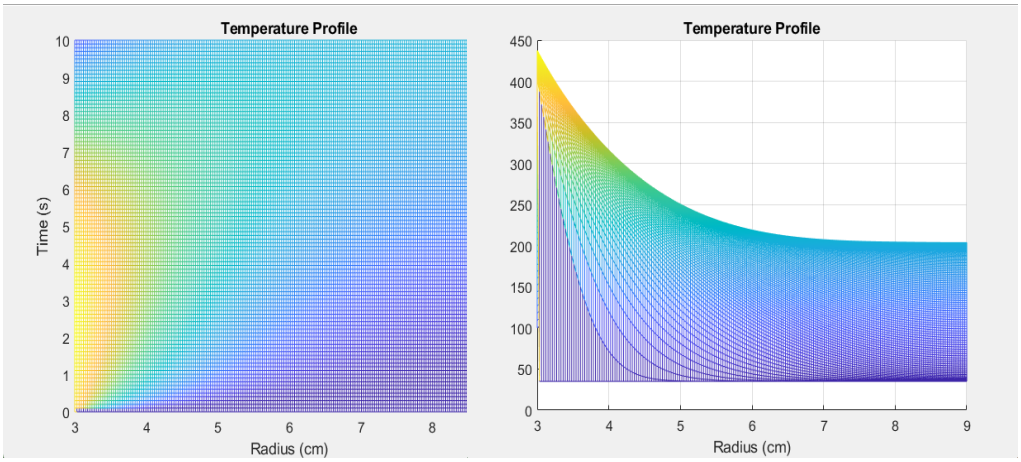
```
    d2Tdr2(i)=(T(i+1)-2*T(i)+T(i-1))./dr.^2;
    dTdr(i)=(T(i+1)-T(i-1))./(2.*dr);
    dTdt(i)=(d2Tdr2(i)+(1./r(i)).*dTdr(i)-n.*(T(i)-Tu))./m;
end
end
```

Listing 1: Solution of Heat Distribution on Circular Fin by Finite Difference Approximation



(a) Temperature Profile Contour (b) Mesh for Temperature Profile

Figure 1: Temperature Profile Contour and Temperature Profile Mesh



(a) X-Y Orientation of Mesh profile (b) X-Z Orientation of Mesh profile

Figure 2: X-Y and X-Z Orientation of Mesh for Temperature Profile

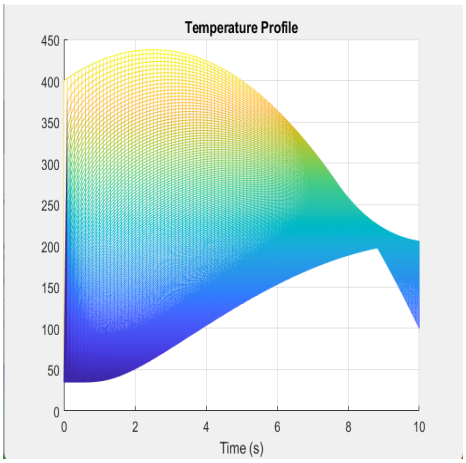


Figure 3: Y-Z Orientation of Mesh profile

2 Solution through MATLAB pdepe Library

The heat distribution on a circular fin is given by the following partial differential equation:

$$\frac{\rho c_p}{k}\frac{dT}{dt} = \frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} - \frac{2h}{kb}(T - T_u) \tag{10}$$

Subjected to intial conditions as:

$$t = 0; r = r; T = T_o \tag{11}$$

and boundary conditions as:

$$r = R_{in}; T(1) = Ts + 30t - 6t^2 \tag{12}$$

$$r = R_{out}; -k\frac{dT}{dr} = h(T - T_u) \tag{13}$$

The general form of solving a partial differential equation using pdepe MATLAB is given as:

$$c(x, t, u, \frac{\partial u}{\partial x})\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}(x^m f(x, t, u, \frac{\partial u}{\partial x})) + s(x, t, u, \frac{\partial u}{\partial x}) \tag{14}$$

where for the heat distribution equation the constants are as

$$c = \frac{\rho c_p}{k}; f = \frac{\partial^2 T}{\partial r^2}; s = \frac{1}{r}\frac{dT}{dr} - \frac{2h}{kb}(T - T_u) \tag{15}$$

By the application of pdepe algorithm on the boundary conditions, the first boundary condition implies:

$$r = R_{in}; T(1) = Ts + 30t - 6t^2 \tag{16}$$

$$(1(x,t,u) - (T_s + 30t - 6t^2)) + 0(x,t)f(x,t,\frac{\partial u}{\partial x}) = 0 \tag{17}$$

and

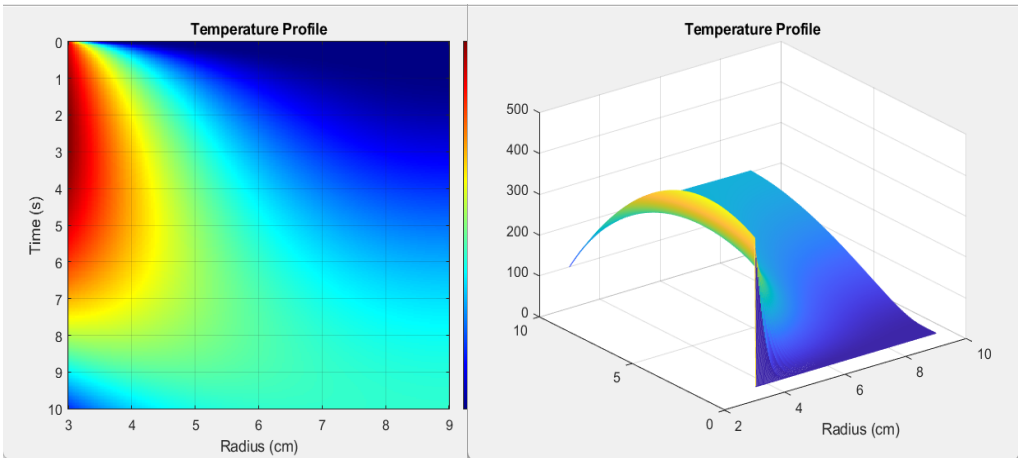
$$r = R_{out}; -k\frac{dT}{dr} = h(T - T_u) \tag{18}$$

$$(h((x,t,u) - Tu)) + k(x,t)f(x,t,\frac{\partial u}{\partial x}) = 0 \tag{19}$$

```
    clc;
clear ;
close all;
%Defining of STep Size and Increment
Rin=3; %Inner Radius (cm)
Rout=9;%Outer Radius (cm)
r=linspace(Rin,Rout,201);
t=linspace(0,10,301); %Time discretization from 1-100 s
m=0;
sol=pdepe(m,@pdefun,@icfun,@bcfun,r,t);
figure(1)
imagesc(r,t,sol)
colormap jet
colorbar
grid on
title('Temperature Profile')
xlabel('Radius (cm)')
ylabel('Time (s)')
figure(2)
mesh (r,t,sol)
title('Temperature Profile')
xlabel('Radius (cm)')
%Building of pdefun
function[c,f,s]=pdefun(r,t,T,dTdr)
%Experimental Data
k=0.8; %Conductivity of Fin (cal/s/cm/K)
h=0.0007;% Heat Transfer Coefficient of Fin (cal/s/cm^2/K)
Tu=30;%Inner Temperature (C)
rho=0.8; %Density of Fin Material (g/cm^3)
cp=0.5; %SpecificHeat Capacity of Fin (cal/g/K)
b=1;
c=rho*cp/k;
f=dTdr;
s=1/r*dTdr-2*h/k/b*(T-Tu);
end
%Input of initial conditions
function T0=icfun(r)
T0=35;
end
%Input of boundary conditions
function [p1,q1,pr,qr]=bcfun(x1,u1,xr,ur,t)
Ts=400;%Surface Temperature (C)
Tu=30;%Inner Temperature (C)
k=0.8; %Conductivity of Fin (cal/s/cm/K)
h=0.0007;% Heat Transfer Coefficient of Fin (cal/s/cm^2/K)
%First Boundary COndition
p1=u1-(Ts+30*t-6*t^2);
q1=0;
```

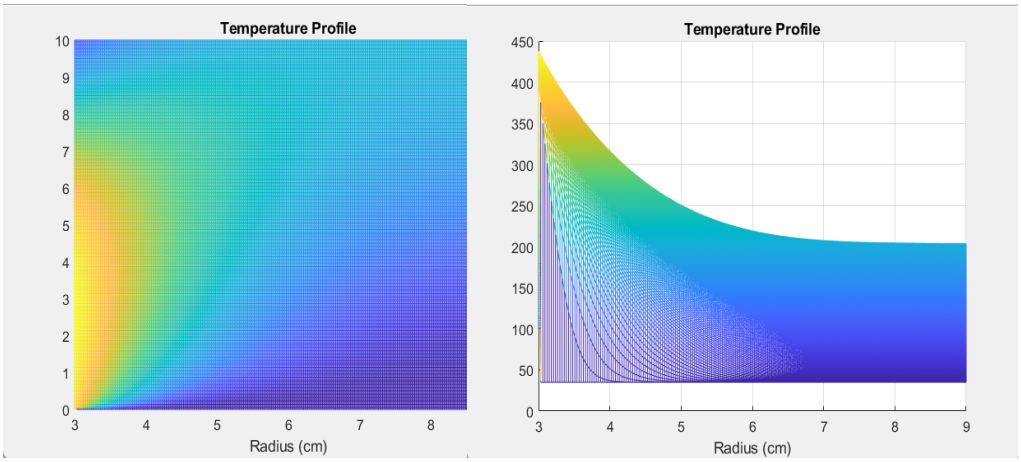
```
%Second Boundary Condition
pr=h*(ur-Tu);
qr=k;
end
```

Listing 2: Solution of Heat Distribution on Circular Fin using pdepe Function



(a) Temperature Profile Contour (b) Mesh for Temperature Profile

Figure 4: Temperature Profile Contour and Temperature Profile Mesh



(a) X-Y Orientation of Mesh profile (b) X-Z Orientation of Mesh profile

Figure 5: X-Y and X-Z Orientation of Mesh for Temperature Profile

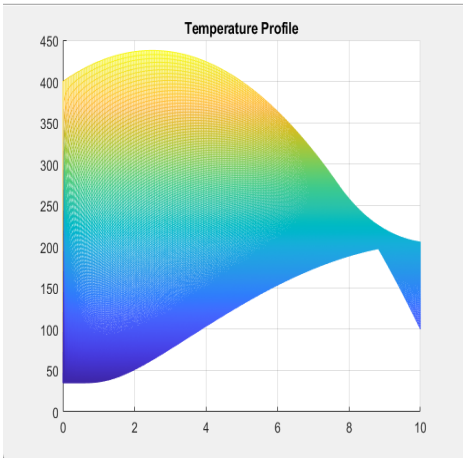


Figure 6: Y-Z Orientation of Mesh profile

3 Conclusion

This study presents a comprehensive analysis of heat distribution on a circular fin using numerical techniques, specifically the finite difference method (FDM) and MATLAB's pdepe solver. The governing heat conduction equation, incorporating both thermal diffusion and convective heat transfer effects, was numerically discretized using first- and second-order finite difference approximations. Additionally, the pdepe function was employed to solve the equation in a generalized form, leveraging MATLAB's built-in partial differential equation solver. The results demonstrate that both approaches effectively model transient heat distribution, capturing temperature variations over time and across the fin's radius. The finite difference method provides a stepwise, structured approach that ensures precision by implementing backward second-order approximations at the boundaries. Meanwhile, the pdepe solver offers a robust and computationally efficient means of solving the partial differential equation, streamlining implementation for engineering applications. The generated temperature profiles and contour plots validate the accuracy of these numerical methods, aligning with theoretical expectations of heat dissipation in circular fins. The findings of this study have significant implications for thermal management applications across various engineering domains, including heat exchanger design, electronic cooling systems, and energy-efficient thermal components. Understanding how heat propagates within finned surfaces is crucial for optimizing cooling performance, enhancing energy efficiency, and improving material selection in high-temperature environments. For future work, this numerical framework can be expanded to incorporate variable material properties, non-uniform heat sources, and radiation effects, which would further enhance the model's predictive capabilities. Additionally, extending the analysis to complex fin geometries, such as perforated or composite fins, could provide deeper insights into real-world

thermal optimization challenges. By refining these computational models, engineers can develop more efficient and effective heat dissipation systems tailored to specific industrial and technological needs.