One Dimensional Convective Heat Transfer Solution for Rotating Flows: Heat Transfer in Circular Couette Flow

Muhammad Abdullah Zafar Ghauri[a]

[a]School of Mechanical and Manufacturing Engineering (SMME), National University of Sciences and Technology (NUST), Islamabad Capital Territory, H-12, 44000, Pakistan

mghauri.ug22smme@student.nust.edu.pk

19th July 2025

Abstract

This study presents an analytical and computational investigation of heat transfer in circular Couette flow with viscous dissipation. The flow involves an incompressible fluid confined between two concentric cylinders, where the inner cylinder rotates at a constant angular velocity and the outer cylinder is stationary and maintained at a fixed temperature. The velocity profile is derived from the Navier-Stokes equations, and the temperature distribution is obtained by solving the energy equation, accounting for viscous heating. The solution is expressed in dimensionless form using the Brinkman number and geometric parameters. An Octave code is provided to visualize the temperature distribution, revealing a significant temperature rise near the inner cylinder due to viscous dissipation. The analysis underscores the importance of viscous heating in high-shear applications such as lubrication and viscometry.

Keywords: Couette flow; viscous dissipation; Navier-Stokes equation; heat transfer; viscometryr; Octave, cylindrical Brinkman Number

Couette flow is the flow of some viscous fluid between two surfaces in space, where one of the surfaces moves tangentially relative to the other. Due to relative motion of surface, shear stress is induced in the surfaces which induces fluid flow. The terminology of Couette flow can be further enhanced for applied pressure gradients in the flow direction. The phenomenon is extensively applied in viscometry and for demonstrating the approximations of reversibility Guyon et al. (2001); Heller (1960). The Couette flow was first considered by Maurice Couette. Couette flow models provide simple heat transfer model for flow between two parallel plates White and Majdalani (2021). The physical importance of the model improves by analyzing over wide variety of geometries, especially in understanding parameters governing velocity and temperature distribution.

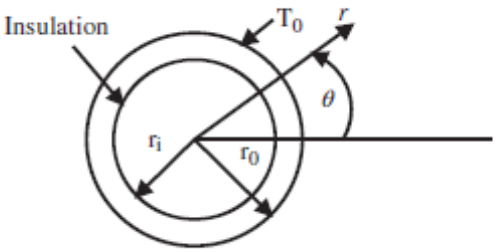


Figure 1: Heat Transfer in Circular Couette Flow in Cylindrical Geometry Onur (2023)

Consider 1, an incompressible hydrodynamically fully developed laminar, steady and asymmetric flow moving through clearance between two infinitely long concentric cylinders. The fluid is an incompressible lubricating oil. The flow associated is referred to as circular Couette flow and serves as the basis for Couette rotational-type viscometers. Associated with the cylindrical geometry are some parameters, such as the inner radius r_i, outer radius r_i, and angular rotational velocity of inner insulated cylinder ω. The outer cylinder is fixed and its temperature is T_o. Considering HFD and viscous dissipation into account, the temperature distribution can be determined

within the lubrication oil.

For effective analysis of phenomenon, following assumptions may be taken into account:

- Flow remains steady throughout, such that $\frac{\partial}{\partial t} = 0$
- The flow rests in laminar regime
- The fluidic properties such as density ρ, heat capacity c_p, conductivity k and viscosity (dynamic, μ) remain constant throughout.
- Fluid flow is assymetric, thus $\frac{\partial}{\partial \theta} = 0$
- The fluid around plates is Newtonian
- The flow is fully developed with the velocity and temperature distribution not changing along flow direction.
- The gravitational effects are ignored and there is no energy generation with the fluid motion
- Negligible axial temperature variation and continuum remains established
- Since the cylinders are infinitely long, thus the end and edge effects are ignored, such that $\frac{\partial}{\partial z} = 0$ and $\frac{\partial}{\partial \theta} = 0$

Due to viscous dissipation, energy is generated in fluid flow, which is removed from fluid through conduction on the pipe surface. The temperature distribution is necessary for determining the Nusselt number. However, determining velocity distribution is crucial. The continuity equation can be expressed as

$$\frac{D\rho}{Dt} + \rho(\vec{\nabla}.\vec{V}) = 0 \tag{1}$$

or

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{2}$$

For an incompressible fluid

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial \theta} = \frac{\partial \rho}{\partial r} = \frac{\partial \rho}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} = 0 \qquad \text{Steady Flow}$$
$$\frac{\partial v_z}{\partial z} = 0 \qquad \text{Pipe is infinitely long, steady, HFD flow}$$
$$\frac{\partial}{\partial \theta} = 0 \qquad \text{Axisymmetric fluid flow}$$

From the above simplifications, the continuity equation reduces to following format:

$$\frac{\partial}{\partial r}(r v_r) = 0$$

On the basis of such simplifications, the continuity equation once integrated results in

$$r v_r = C$$

By the no-slip boundary condition, since it is applied at the walls of pipe surfaces to determine the constant of integration $f(z)$

$$r = r_o \qquad v_r = 0 \qquad \text{Impermeable wall}$$

which implies $C = 0$, so the radial component of velocity becomes

$$v_r(r, z) = 0$$

which further implies that the streamlines are concentric in flow field. The radial component of velocity v_r is zero everywhere. Considering the θ-component of Navier-Stokes equation, which can be sourced from the momentum equation

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z}\right) = \rho f_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right)$$

$$f_\theta = \rho f_\theta = 0 \qquad \text{No effects of gravity}$$
$$\frac{\partial v_\theta}{\partial t} = 0 \qquad \text{Steady flow}$$
$$\frac{\partial v_\theta}{\partial z} = 0 \qquad \text{Ignoring end effects, HFD steady flow}$$
$$v_r \frac{\partial v_\theta}{\partial r} = 0 \text{ since } v_r = 0 \qquad \text{Equation of continuity}$$
$$v_z \frac{\partial v_\theta}{\partial z} = 0 \text{ since } \frac{\partial v_z}{\partial z} = 0 \qquad \text{Infinite long pipe, HFD steady fluid flow}$$
$$\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} = 0, \text{ since } \frac{\partial v_\theta}{\partial \theta} = 0 \qquad \text{Axisymmetric flow}$$
$$\frac{\partial v_\theta}{\partial \theta} = 0 \text{ since } v_\theta = 0 \qquad \text{Axisymmetric flow}$$
$$\frac{\partial^2 v_\theta}{\partial z^2} = 0 \text{ since} \frac{\partial v_\theta}{\partial z} = 0 \qquad \text{Infinitely long tube, steady HFD flow}$$

The θ-component of momentum equation reduces to

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) = 0$$

The correspondnig boundary conditions are:

$$v_\theta(r_1) = \omega r_1$$
$$v_\theta(r_0) = 0$$

Once plugged in MATLAB solver using the following code, the resultant expression becomes:

```
clear all
clc

syms r v(r) r0 r1 omega

% Define the differential equation from the image
de = diff( (1/r) * diff(r * v, r), r ) == 0;

% Define boundary conditions
BC1 = v(r1) == omega * r1;
BC2 = v(r0) == 0;
BC = [BC1, BC2];

% Solve the differential equation
sol = dsolve(de, BC);
sol=expand(sol)
```

Listing 1: MATLAB code for evaluating the velocity profile

The general integration of velocity profile for cylindrical profile is given as

$$v_\theta = \frac{C_1}{2} r + \frac{C_2}{r}$$

where C_1 and C_2 are constants of integration, evaluated in MATLAB code subjected to boundary conditions:

$$C_1 = -\frac{2\omega r_1^2}{r_o^2 - r_1^2}$$

and

$$C_2 = \frac{\omega r_1^2 r_o^2}{r_o^2 - r_1^2}$$

The MATLAB code results in following expression:

$$v_\theta = \frac{\omega\, {r_0}^2\, {r_1}^2}{r\, {r_0}^2 - r\, {r_1}^2} - \frac{\omega\, r^2\, {r_1}^2}{r\, {r_0}^2 - r\, {r_1}^2}$$

The final expression can be expressed as

$$v_\theta = \frac{r}{2}\left[-\frac{2\omega r_1^2}{r_o^2 - r_1^2}\right] + \frac{1}{r}\left[\frac{\omega r_1^2 r_o^2}{r_o^2 - r_1^2}\right]$$

or

$$\frac{v_\theta(r)}{\omega r_1} = \frac{\left(\frac{r_0}{r_1}\right)^2\left(\frac{r_1}{r}\right) - \left(\frac{r}{r_1}\right)}{\left(\frac{r_0}{r_1}\right)^2 - 1}$$

From the velocity distribution once known, the energy equation in cylindrical coordinates coupled with viscous dissipation can be taken into consideration:

$$\rho c_p\left[\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right] = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right] + q''' + \mu\Phi$$

where

$$\Phi = 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{v_\theta}{\partial z}\right]^2 + \left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]^2$$

$q''' = 0$ No energy generation

$\frac{\partial T}{\partial t} = 0$ Steady Flow

$v_r\frac{\partial T}{\partial r} = 0$ Since $v_r = 0$ due to continuity equation

$v_z\frac{\partial T}{\partial z} = 0$ No end effects

$\frac{v_\theta}{r}\frac{\partial T}{\partial \theta} = 0$ since $\frac{\partial T}{\partial \theta} = 0$ Axisymmetric flow

$\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$ since $\frac{\partial T}{\partial \theta} = 0$ Axisymmetric flow

$\frac{\partial T}{\partial z} = 0$ No edge effect and w=0 since there is no motion in z-direction

$\frac{\partial^2 T}{\partial z^2} = 0$ Negligible axial conduction

From all the following aspects, the energy equation reduces to the following expression

$$\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \mu\Phi = 0$$

Considering the assumptions made, the viscous dissipation term reduces to

$$\Phi = \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]^2$$

=

$$\left[\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right]^2$$

By substituting the velocity distribution into viscous dissipation terms results in

$$\Phi = \frac{4\omega^2 r_i^4 r_o^4}{(r_o^2 - r_i^2)r^4}$$

or

$$\Phi = \left(\frac{2\omega r_i^2}{1 - \left(\frac{r_i}{r_o}\right)^2}\right)^2 \frac{1}{r^4}$$

The viscous dissipation term once substituted in the energy equation results in following expression:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\frac{\mu}{k}\left(\frac{2\omega r_i^2}{1 - \left(\frac{r_i}{r_o}\right)^2}\right)^2 \frac{1}{r^4}$$

The corresponding boundary conditions are

$$\frac{dT}{dr}(r_i) = 0$$
$$T(r_o) = T_o$$

```
clear all
clc
syms r  v(r) mu dp dz R V T(r) TW k

%Temperature Distribution
v=2*V*(1-(r/R)^2)
F=(diff(v,r))^2
de1=r*diff(T,r,2)+diff(T,r)==-(mu*r/k)*F

DT=diff(T,r)
BC1=T(0)==0
BC2=T(R)==TW

BC=[BC1,BC2]

sol=dsolve(de1, BC)
```

Listing 2: MATLAB code for evaluating temperature profile

On running the above MATLAB following equation is obtained

$$T = -\frac{\mu}{4k}\left(\frac{2\omega r_i^2}{1-\left(\frac{r_i}{r_o}\right)^2}\right)^2 \frac{1}{r^2} + C_3 ln(r) + C_4$$

Where the unknown constants C_3 and C_4 are obtained from boundary conditions

$$C_3 = -\frac{\mu}{2k}\left(\frac{2\omega r_i^2}{1-\left(\frac{r_i}{r_o}\right)^2}\right)^2 \frac{1}{r_i^2}$$

$$C_4 = T_o + \frac{\mu}{4k}\left(\frac{2\omega r_i^2}{1-\left(\frac{r_i}{r_o}\right)^2}\right)^2 \left[\frac{1}{r_o^2} + \frac{2}{r_i^2} ln(r_o)\right]$$

These constants can be plugged in temperature distribution equation such that

$$T(r) = T_o + \frac{\mu}{4k}\left(\frac{2\omega r_i^2}{1-\left(\frac{r_i}{r_o}\right)^2}\right)^2 \left[\left(\frac{r_i}{r_o}\right)^2 - \left(\frac{r_i}{r}\right) + 2ln\left(\frac{r_o}{r}\right)\right]$$

In dimensionless form, the equation cen be expressed as

$$\frac{T(r) - T_o}{\frac{\mu}{4k}\left(\frac{2\omega r_i^2}{1-\left(\frac{r_i}{r_o}\right)^2}\right)^2} = \left[\left(\frac{r_i}{r_o}\right)^2 - \left(\frac{r_i}{r}\right) + 2ln\left(\frac{r_o}{r}\right)\right]$$

The maximum temperature can be evaluated at the shaft's surface, setting r=r_i, following expression can be obtained

$$T(r_i) - T_o = \frac{\mu}{4k}\left(\frac{2\omega r_i^2}{1-\left(\frac{r_i}{r_o}\right)^2}\right)^2 \left[\left(\frac{r_i}{r_o}\right)^2 - 1 + 2ln\left(\frac{r_o}{r}\right)\right]$$

Since dimenionless plot is not possible due to tedious reductions, Octave can be used for plotting curve for temperature distribution. The physical parameters can be randomly defined and plugged to evaluate temperature profile. The Octave code is given as

```
    % Parameters
r_i = 0.1;        % Inner radius [m]
r_o = 0.2;        % Outer radius [m]
T_o = 300;        % Outer wall temperature [K]
mu = 0.1;         % Dynamic viscosity [Pa s]
k = 0.15;         % Thermal conductivity [W/(m K)]
omega = 100;      % Angular velocity [rad/s]

% Calculate characteristic velocity
kappa = r_i / r_o;
V_c = (2 * omega * r_i) / (1 - kappa^2);

```

```
% Precompute constant factor
constant_factor = (mu / (4 * k)) * V_c^2;

% Radial positions (from r_i to r_o)
r = linspace(r_i, r_o, 100);

% Dimensionless radial coordinate
eta = r / r_o;

% Calculate temperature distribution
T = T_o + constant_factor * (...
    kappa^2 * (1 - 1./eta.^2) - 2 * log(eta)...
);

% Plot results
figure;
plot(r, T, 'b-', 'LineWidth', 2);
hold on;
grid on;

% Mark key points
plot(r_i, T(1), 'ro', 'MarkerSize', 8, 'MarkerFaceColor', 'r');   % Inner wall
plot(r_o, T_o, 'go', 'MarkerSize', 8, 'MarkerFaceColor', 'g');    % Outer wall

% Labels and title
xlabel('Radial Position, r [m]');
ylabel('Temperature, T(r) [K]');
title('Temperature Distribution in Taylor-Couette Flow with Viscous Dissipation');
legend('Temperature Profile', 'Inner Wall', 'Outer Wall', 'Location', 'best');

% Add parameter info to plot
param_text = {
    sprintf('r_i = %.2f m', r_i),
    sprintf('r_o = %.2f m', r_o),
    sprintf('T_o = %d K', T_o),
    sprintf('\\mu = %.2f Pa\\cdots', mu),
    sprintf('k = %.2f W/(m\\cdotK)', k),
    sprintf('\\omega = %d rad/s', omega)
};
text(0.15, max(T)-10, param_text, 'BackgroundColor', 'white');
```

Listing 3: Temperature Distribution Code

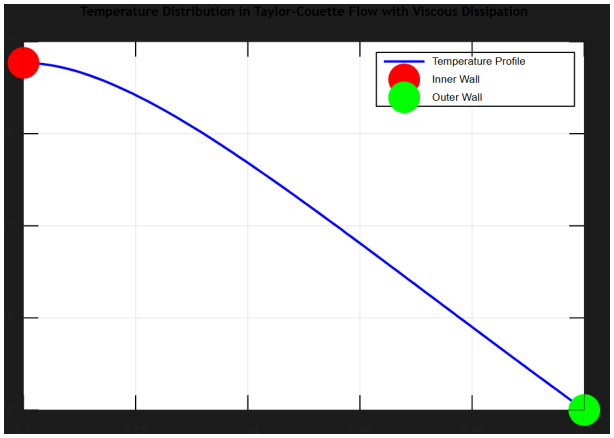


Figure 2: Temperature Distribution about the surface

Conclusion

The analysis of heat transfer in circular Couette flow with viscous dissipation provides valuable insights into the thermal behavior of fluids in rotating systems. The derived temperature profile, expressed in terms of the Brinkman number and geometric ratios, shows that the maximum temperature occurs at the inner cylinder surface, highlighting the critical role of viscous heating in high-shear regions. The Octave implementation effectively

visualizes this distribution, aiding in the understanding of heat transfer mechanisms. This work has practical implications for the design of rotating machinery, lubrication systems, and viscometers.

References

Guyon, E., Hulin, J.-P., Petit, L., & Mitescu, C. D. (2001). *Physical hydrodynamics*. Oxford University Press.

Heller, J. P. (1960). An unmixing demonstration. *American Journal of Physics*, *28*(4), 348–353. DOI: 10.1119/1.1935802

Onur, N. (2023). *Introduction to convective heat transfer a software-based approach using maple and matlab*. Wiley: John Wiley Sons. DOI: 9781119766766

White, F. M., & Majdalani, J. (2021). *Viscous fluid flow* (4th ed.). New York: McGraw Hill.