Analytical and Numerical Solution of Burger's Equation

Muhammad Abdullah Zafar Ghauri

7th April 2025

Abstract

This study effectively focuses on comprehensive investigation of the Burger's Equation by formulating analytically and numerically the solutions for the one-dimensional viscous form of Burger's Equation with Dirichlet boundary conditions and sinusoidal initial conditions. The numerical approach is adopted in finite difference discretization in MATLAB, thereby transforming the convection-diffusion dynamics into a solvable system of nonlinear ODEs. The analytical solution is constructed via a Fourier series representation involving Bessel-function-like damping terms and exponential decay. Through detailed contour and surface plots of velocity profiles, the investigation for the solution dynamics and evaluate convergence across varying time instances is carried out effectively. The results demonstrate alignment between numerical and analytical methods, highlighting the role of viscosity and initial spectral content in the evolution of flow structures.

Keywords: Dirichlet boundary Conditions, Fourier Series, Bessel Function, exponential decay, sinusoidal conditions

1 Introduction

Burger's Equation, also referred to as Bateman-Burgers Equation, is a fundamental partial differential equation and convection-diffusion equation [3] extensively used in applied sciences such as fluid mechanics, nonlinear aucostics, gas dynamics and traffic flow[5], originally developed by Harry Bateman in 1915 [1, 6] and later analyzed by Johannes Martinus Burgers in 1948[2]. Burger's Equation is an elliptical partial differential equation, generally expressed as for viscous fluid in one-dimensional space :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{1}$$

In absence of viscosity (which implies ν=0), the Burger's equation is reduced to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{2}$$

In generalized form, the Burgers Equation for more than two dimensions can be expressed as:

$$\frac{\partial u}{\partial t} + u.\nabla u = \nu\nabla^2 u \tag{3}$$

The Burger's Equation can be expressed in much generalized form where it extends the quasilinear convective to more generalized form as:

$$\frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{4}$$

Where c(u) is the arbitrary function of u and the inviscid scenario i.e $\nu = 0$ equation is still a quasilinear hyperbolic equation for $c(u) > 0$ and its solution can be constructed using method of characteristics as before.

2 Mathematical Model for Numeric Solution

The Burger's Equation is defined as 1 can be modified as:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x}$$

Subjected to boundary conditions:

$$u(0,t) = 0 \qquad u(1,t) = 0$$

and initial condition as:

$$u(x,0) = sin(\pi x)$$

For the very specific problem as defined in [4], first spatial derivatives are approximated using finite difference method, which leads to the transformation of nonlinear partial differential equation into a system of ordinary differential equation of nonlinear nature.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} = \nu \frac{u(i+1) + u(i-1) - 2u(i)}{\Delta x^2} - u \frac{u(i+1) - u(i-1)}{2\Delta x} \tag{5}$$

```
    clc;
clear;
close all;

%Experimental Data
nu=0.1; %Kinematic viscosity

%Defining of time step
Nt=3001;
t=linspace(0,2,Nt);
dt=t(2)-t(1);

%Defining of discretization from mesh
Nx=1001;
x=linspace(0,1,Nx);
dx=x(2)-x(1);

%Defining of initial conditions
IC=ones(1,Nx).*sin(pi.*x);

%Solution using ode15s library
[t u]=ode15s(@fpde,t,IC,[],nu,dx,Nx);

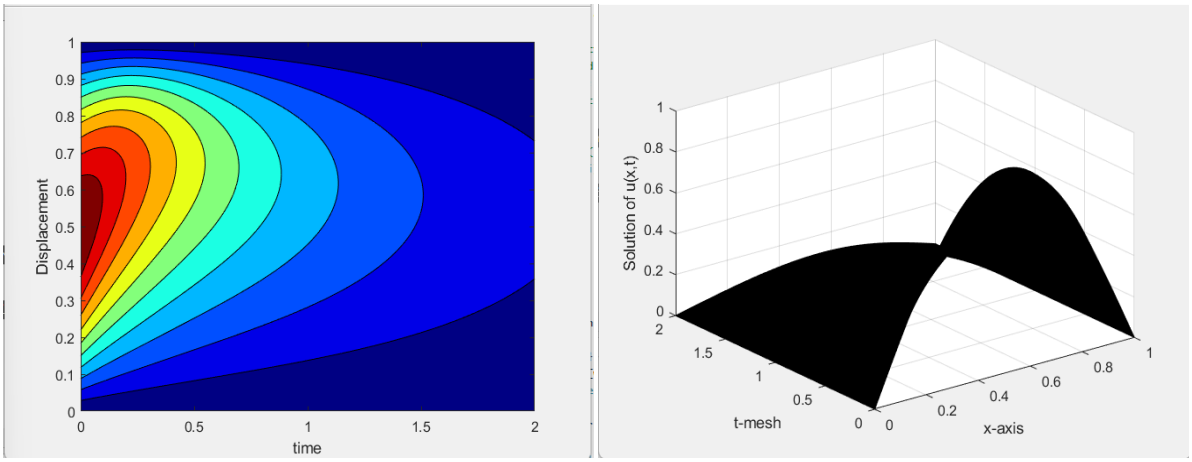
%Plotting of solution
figure(1)
contourf(t,x,u',10)
colormap jet
xlabel('time')
ylabel('Displacement')

figure(2)
surf(x,t,u)
xlabel('x-axis')
ylabel('t-mesh')
zlabel('Solution of u(x,t)')

%Defining of function
function dudt=fpde(t,u,nu,dx,Nx)
dudt=zeros(Nx,1); %Defininng of preallocation matrix

for i=2:Nx-1
    dudt(i)=nu.*(u(i+1)-2*u(i)+u(i-1))./dx^2-u(i).*(u(i+1)-u(i-1))./2./dx;
end
end
```

Listing 1: MATLAB code for numerical evalutaion of Burger's Equation



(a) Velocity Contour (b) Surface Mesh

Figure 1: Contour for Velocity Distribution and Surface Mesh

3 Mathematical Model for Analytical Solution

The Burger's Equation is defined as 1 can be modified as:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x}$$

Subjected to boundary conditions:

$$u(0,t) = 0 \qquad u(1,t) = 0$$

and initial condition as:

$$u(x,0) = sin(\pi x)$$

The analytical solution as provided in [4] is given as:

$$u(x,t) = 2\pi\nu \frac{\sum_{n=1}^{\infty} C_n \exp(-n^2\pi^2\nu t) \cdot n \sin(n\pi x)}{C_0 + \sum_{n=1}^{\infty} C_n \exp(-n^2\pi^2\nu t) \cos(n\pi x)} \tag{6}$$

where

$$C_0 = \int_0^1 \exp\left\{-\frac{1}{2\pi\nu}\left[1 - \cos(\pi x)\right]\right\} dx,$$

$$C_n = 2\int_0^1 \exp\left\{-\frac{1}{2\pi\nu}\left[1 - \cos(\pi x)\right]\right\} \cos(n\pi x)\, dx.$$

```
    clc;
clear;
close all;

%Experimental Data
nu=0.1; %Kinematic viscosity

%Defining of time step
Nt=3001;
t=linspace(0,2,Nt);
dt=t(2)-t(1);

%Defining of discretization from mesh
Nx=1001;
x=linspace(0,1,Nx);
dx=x(2)-x(1);

%Defining of initial conditions
```

```
IC=ones(1,Nx).*sin(pi.*x);

%Solution using ode15s library
[t u]=ode15s(@fpde,t,IC,[],nu,dx,Nx);
%Computation of Analytical Solution
N_terms = 100;
u_ana = zeros(Nt, Nx);
for ti = 1:Nt
    u_ana(ti, :) = analytical_solution(x, t(ti), nu, N_terms);
end

%Plotting of solution
figure('Name', 'Numerical Solution - Contour')
contourf(t_sol, x, u_num', 10)
colormap jet
xlabel('Time')
ylabel('x')
title('Numerical Solution (Contour)')

figure('Name', 'Numerical Solution - Surface')
surf(x, t_sol, u_num)
xlabel('x')
ylabel('Time')
zlabel('u(x,t)')
title('Numerical Solution (Surface Plot)')

figure('Name', 'Analytical Solution - Contour')
contourf(t, x, u_ana', 10)
colormap parula
xlabel('Time')
ylabel('x')
title('Analytical Solution (Contour)')

figure('Name', 'Analytical Solution - Surface')
surf(x, t, u_ana)
xlabel('x')
ylabel('Time')
zlabel('u(x,t)')
title('Analytical Solution (Surface Plot)')

% Times to compare (in seconds)
compare_times = [0, 0.5, 1, 2];
figure('Name', 'Comparison of Analytical and Numerical Solutions');

for i = 1:length(compare_times)
    t_target = compare_times(i);

    % Find the closest time index
    [~, tidx] = min(abs(t - t_target));

    % Extract solutions at selected time
    u_num_t = u_num(tidx, :);
    u_ana_t = u_ana(tidx, :);

    subplot(2,2,i)
    plot(x, u_num_t, 'b-', 'LineWidth', 1.5)
    hold on
    plot(x, u_ana_t, 'r--', 'LineWidth', 1.5)
    title(['u(x,t) at t = ' num2str(t(tidx), '%.2f') ' s'])
    xlabel('x')
    ylabel('u(x,t)')
    legend('Numerical', 'Analytical')
    grid on
end
%Defining of function
function dudt=fpde(t,u,nu,dx,Nx)
dudt=zeros(Nx,1); %Defininng of preallocation matrix

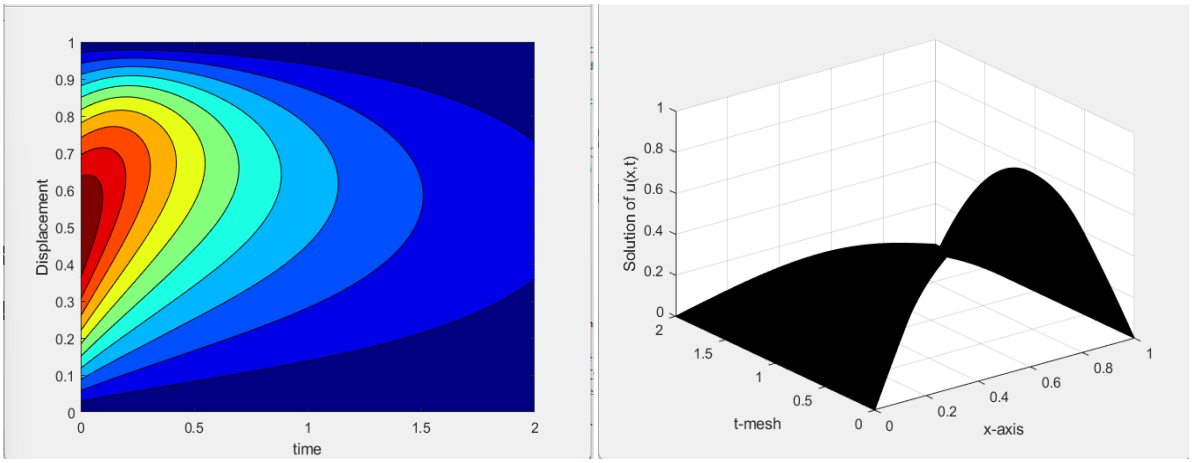
for i=2:Nx-1
    dudt(i)=nu.*(u(i+1)-2*u(i)+u(i-1))./dx^2-u(i).*(u(i+1)-u(i-1))./2./dx;
end
```

```
end
% Analytical solution
function u_exact = analytical_solution(x, t, nu, N)
    C0 = integral(@(x_val) exp(-(1/(2*pi*nu)) * (1 - cos(pi*x_val))), 0, 1);
    numerator = zeros(size(x));
    denominator = C0 * ones(size(x));

    for n = 1:N
        Cn = 2 * integral(@(x_val) exp(-(1/(2*pi*nu)) * (1 - cos(pi*x_val))) .* ...
             cos(n*pi*x_val), 0, 1);
        En = exp(-n^2 * pi^2 * nu * t);
        numerator = numerator + Cn * En * n * sin(n*pi*x);
        denominator = denominator + Cn * En * cos(n*pi*x);
    end

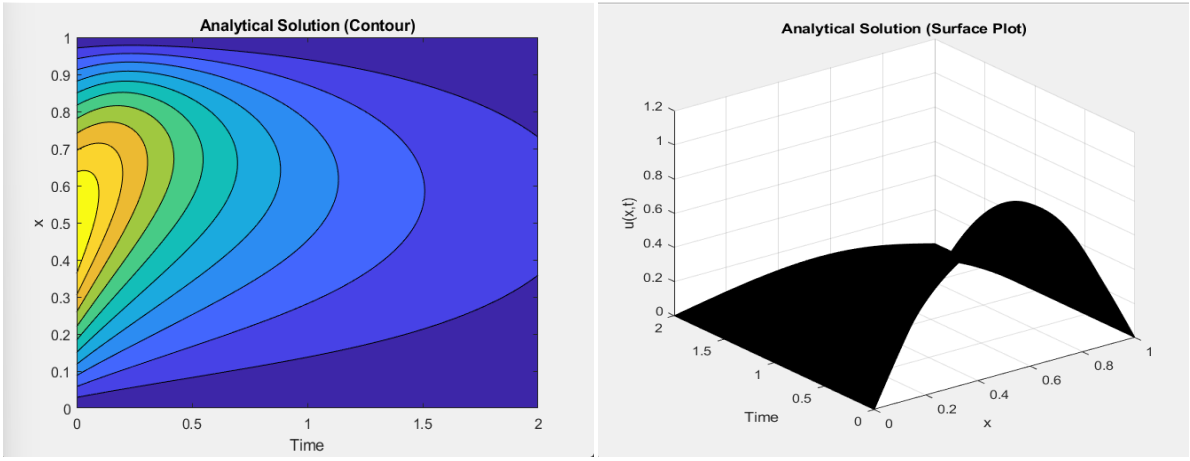
    u_exact = 2 * pi * nu * numerator ./ denominator;
end
```

Listing 2: MATLAB code for numerical and analytical solution for Burger's Equation



(a) Numerical Solution Velocity Contour (b) Numerical Solution Surface Mesh

Figure 2: Numerical Solution Contour for Velocity Distribution and Surface Mesh



(a) Analytical Solution Velocity Contour (b) Analytical Solution Surface Mesh

Figure 3: Analytical Solution Contour for Velocity Distribution and Surface Mesh

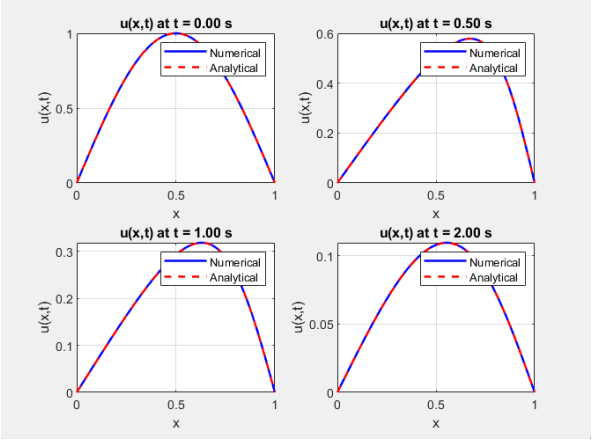


Figure 4: Comparison for Analytical and Numerical Solution

Conclusion

This study effectively and efficiently focused on the comparative study between numerical and analytical solutions of the one-dimensional viscous Burger's Equation subjected to Dirichlet boundary conditions and sinusoidal initial conditions with the transformation of the nonlinear PDE into a discretized system using finite difference approximations having enabled an effective numerical simulation of transient dynamics. The analytical method, derived from modal expansion with time-damped Fourier modes, effectively captured the dissipative and convective characteristics of the system. From the graphical results, particularly the velocity contour maps and surface plots, highlight the time-dependent evolution of the flow field, demonstrating the interplay between diffusion and nonlinearity. The alignment of numerical and analytical results validates the consistency and accuracy of both solution frameworks. Moreover, the oscillatory decay of higher modes in the analytical series highlights the role of viscosity in smoothing initial discontinuities or steep gradients, a hallmark of realistic fluid phenomena.

References

[1] Bateman, H. Some recent researches on the motion of fluids. *Monthly Weather Review 43*, 4 (1915), 163–170.

[2] Burgers, J. M. A mathematical model illustrating the theory of turbulence. *Advances in Applied Mechanics 1* (1948), 171–199.

[3] Misra, S., Raghurama Rao, S. V., and Bobba, M. K. Relaxation system based sub-grid scale modelling for large eddy simulation of burgers' equation. *International Journal of Computational Fluid Dynamics 24*, 8 (2010), 303–315.

[4] Mukundan, V., and Awasthi, A. Linearized implicit numerical method for burgers' equation. *Nonlinear Engineering 5*, 4 (2016), 219–234.

[5] Musha, T., and Higuchi, H. Traffic current fluctuation and the burgers equation. *Japanese Journal of Applied Physics 17*, 5 (1978), 811.

[6] Whitham, G. B. *Linear and Nonlinear Waves*, vol. 42. John Wiley & Sons, 2011.