Similarity Solution for Velocity Boundary Layer over a Semi-Infinite Flat Plate

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20th July 2025

Abstract

This document presents a comprehensive study on the similarity transformation approach to solve the laminar boundary layer flow over a semi-infinite vertical flat plate maintained at constant wall temperature. By introducing a similarity variable, the governing partial differential equations (PDEs) are reduced to a third-order nonlinear ordinary differential equation (ODE), known as the Blasius equation. The solution, obtained numerically using Maple 2022, demonstrates the development of the velocity boundary layer and allows computation of important parameters such as displacement thickness, momentum thickness, wall shear stress, and local/average skin friction coefficients. The numerical velocity profiles are validated against experimental data from Junkhan and Seroyvy (1966, 1967), showing strong agreement and confirming the accuracy of the similarity solution. This work reinforces the utility of similarity methods in simplifying complex transport problems and provides insight into viscous effects near solid boundaries under forced convection.

The idea of similarity transformation is to transform a complex partial differential equation to an ordinary differential equations by combining the variables in a special manner. The transformation by similarity variable is typically successful provided the similarity variable remains an independent variable. Through transformation, the number of independent variables are reduced by one, and hence for said reason, two of the original boundary conditions reduce to one.

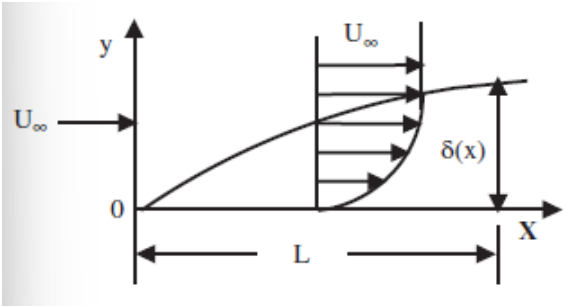


Figure 1: Steady boundary layer flow over a flat plate Onur (2023)

Consider 1, when the fluid flows over the surface, a boundary layer slowly grows into a thin region near the walls, and hence the effect of fluid viscosity is confined to this particular region. The concept of boundary layers was initiated by L. Prandtl in 1904. The momentum boundary layer thickness $\delta(x)$ is typically defined as the distance from wall surface where the velocity u(x,y) reaches 99 % of the freestream velocity U_∞. Boundary layer thickness $\delta(x)$ is the only physically significant length scale whose direction is normal to wall surface. The streamwise length L for advection is much larger than the characteristic length $\delta(x)$ in the direction normal to wall surface. Since $\delta << L$, the diffusion across the boundary layer is much larger than the axial diffusion. In forced convection, the boundary layer thickness does not really influences the exposed fluid flow over flat surface.

Consider the viscous incompressible constant property parallel flow over a sharp-edge thin smooth semi-infinite flat plate, where the problem is a classic interpretation of flow over a semi-infinite flat plate resting at 0°angle of incidence to the uniform stream velocity U_∞ as inferred from 1. The upstream velocity is

uniform and parallel to the flat plate. Assuming thin plates enough not to disturb the uniform exterior flow such that the fluid flows at the constant velocity U_∞ over flat plate. For constant velocity U_∞, $-\left(\frac{1}{\rho}\right)\left(\frac{\partial p}{\partial x}\right) = 0$. Flow is assumed to be steady and two-dimensional along the constant fluid properties. About the edge of boundary layer, the velocity layer is U_∞. The Reynolds number is large enough such that the boundary layer assumptions are valid. Beyond these assumptions, the governing equations are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

The momentum equation in horizontal direction is given as:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

The boundary conditions for momentum equation are as follows:

$$\begin{aligned} u(x,0) &= 0 \qquad \text{No Slip} \\ v(x,0) &= 0 \qquad \text{Solid wall} \\ u(x,\infty) &\to U_\infty \\ u(0,y) &\to U_\infty \end{aligned}$$

The said problem was solved by Blasius in 1908 using similarity transformation Evans (1968). Considering

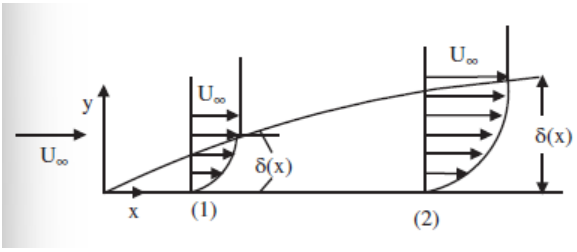


Figure 2: Similar cubic velocity profiles Onur (2023)

the very problem, there is no characteristic length in any direction. Velocity profiles in boundary layer are expected to be similar for all values of x starting from the leading edge of plate, as shown in 2. That is, u(x_1,y) at location 1, similar to u(x_2,y) at location 2, assuming that y-coordinate is stretched by change in boundary layer thickness $\delta(x)$. Hence, it can be implies that despite the shapes of velocity profiles are geometrically similar at different x-positions along the plate, which only differ by stretching factor along y-direction despite that despite the growth of the boundary layer with distance x from the leading edge. Assuming x-component of velocity having similar profile for all values of x positions along the flat plate for any fluid flowing with any particular freestream velocity U_∞. The velocity profiles confined within the boundary layer can be conveniently discussed in terms of dimensionless variables, as shown in 3. The y distance is generally normalized with the local boundary layer thickness $\delta(x)$ while the local velocity component is normalized with freestream velocity U_∞. It may be concluded that the dimensionless velocity profile shown in 3 is same for all laminar flows along the flat plate for all positions on the flat plate. The similarity of profiles implies the existence of a unique function such that

$$\frac{u}{U_\infty} = f_1\left(\frac{y}{\delta}\right) \tag{3}$$

Where $\delta(x)$ is the local boundary layer thickness. Apart from being very close to the wall, the velocity u is of the order of freestream velocity U_∞ and the distance y is measured from the plate wall is of the order of boundary layer thickness $\delta(x)$. Hence

$$u \sim U_\infty \tag{4}$$

$$y \sim \delta \tag{5}$$

Using 4 and 5, the continuity equation becomes

$$\frac{U_\infty}{x} + \frac{v}{\delta} \sim 0 \tag{6}$$

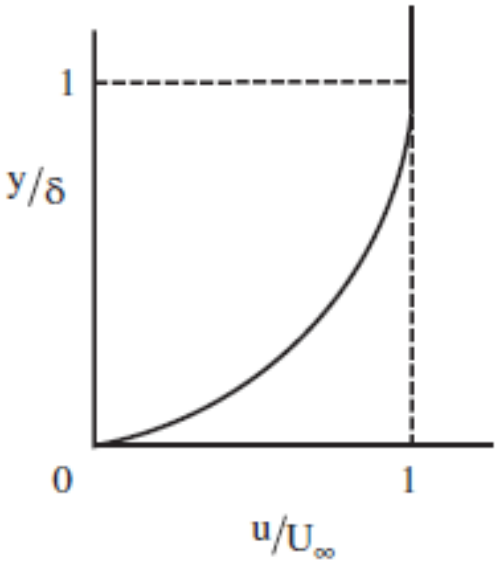


Figure 3: Dimensionless velocity profile Onur (2023)

or

$$v \sim \frac{\delta U_\infty}{x} \tag{7}$$

Using the above order of magnitude terms in momentum equation, the following expression is deduced

$$U_\infty \frac{U_\infty}{x} + \left(\frac{U_\infty}{x}\right)\left(\frac{U_\infty}{\delta}\right) \approx \nu \left(\frac{U_\infty}{\delta^2}\right) \tag{8}$$

or

$$\delta \sim \sqrt{\frac{\nu x}{U_\infty}} \tag{9}$$

9 can be expressed in non dimensional form as

$$\frac{\delta}{x} \sim \sqrt{\frac{\nu}{U_\infty x}} = \frac{1}{\sqrt{Re_x}} \tag{10}$$

where $Re_x = \frac{xU_\infty}{\nu}$ is the local Reynolds number. Thus, the similarity principles become

$$\frac{u}{U_\infty} = f_2\left(y\sqrt{\frac{U_\infty}{\nu x}}\right) \tag{11}$$

where the parenthesis term is a dimensionless coordinate. It is consistent with the experimental measurements made with momentum boundary layer flows. The similarity variables for the concerned problem can be obtained through methodology described in Arpaci and Larsen (1984). By choosing arbitrary selected reference lengths x_o and y_o, the inherent characteristic velocity U_∞, and the arbitrary selected reference velocity V_o. Following dimensionless quantities can be defined

$$x^* = \frac{x}{x_o} \tag{12}$$

$$y^* = \frac{y}{y_o} \tag{13}$$

$$u^* = \frac{u}{U_\infty} \tag{14}$$

$$v^* = \frac{v}{V_o} \tag{15}$$

Using 12, 13, 14, and 15, the equations of continuity and momentum can effectively be non-dimensionalized. Following noondimensionalization, the continuity and momentum equations can be expressed as:

$$\left(\frac{U_\infty y_o}{V_o x_o}\right)\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0 \tag{16}$$

$$\left(\frac{U_\infty y_o^2}{\nu x_o}\right)u^*\frac{\partial u^*}{\partial x^*}+\left(\frac{V_o y_o}{\nu}\right)v^*\frac{\partial v^*}{\partial y^*}=\frac{\partial^2 u^*}{\partial y^{*2}} \tag{17}$$

This further implies that the non-dimensionalized velocity ratios can be expressed as functions of various properties

$$\frac{u}{U_\infty}=f_1\left(\frac{x}{x_o},\frac{y}{y_o},\frac{V_o y_o}{\nu},\frac{U_\infty y_o}{V_o x_o},\frac{U_\infty y_o^2}{\nu x_o}\right) \tag{18}$$

$$\frac{v}{V_o}=g_1\left(\frac{x}{x_o},\frac{y}{y_o},\frac{V_o y_o}{\nu},\frac{U_\infty y_o}{V_o x_o},\frac{U_\infty y_o^2}{\nu x_o}\right) \tag{19}$$

The reference quantities, x_o, y_o, and V_o are not characteristic properties. Thus, they can be eliminated step by step. Mathematical principles permit the transformation of variables, yet the changing of variables is not permitted. The procedure can be initiated by elimination of V_o. 18 and 19 can be transformed such that only one term dependent on V_o remains, A new parameter shall be introduced for $\frac{U_\infty y_o}{V_o x_o}$, which is obtained through multiplication of $\frac{U_\infty y_o}{V_o x_o}$ by $\frac{V_o y_o}{\nu}$. Thus,

$$\left(\frac{U_\infty y_o}{V_o x_o}\right)\left(\frac{V_o y_o}{\nu}\right)=\left(\frac{U_\infty y_o^2}{\nu x_o}\right)$$

18 and 19 has the above parameter $\left(\frac{U_\infty y_o^2}{\nu x_o}\right)$, therefore the mathematical expression becomes:

$$\frac{u}{U_\infty}=f_2\left(\frac{x}{x_o},\frac{y}{y_o},\frac{V_o y_o}{\nu},\frac{U_\infty y_o^2}{\nu x_o}\right) \tag{20}$$

Multiplying $\frac{V_o y_o}{\nu}$ by left hand side of 19, the expression becomes

$$\left(\frac{V_o y_o}{\nu}\right)\left(\frac{v}{V_o}\right)=\frac{v y_o}{\nu}$$

Thus, 19 takes shape as

$$\frac{v y_o}{\nu}=g_2\left(\frac{x}{x_o},\frac{y}{y_o},\frac{V_o y_o}{\nu},\frac{U_\infty y_o^2}{\nu x_o}\right) \tag{21}$$

20 and 21 have physical significance in u and v provided them being independent of V_o. Hence

$$\frac{u}{U_\infty}=f_2\left(\frac{x}{x_o},\frac{y}{y_o},\frac{U_\infty y_o^2}{\nu x_o}\right) \tag{22}$$

$$\frac{v y_o}{\nu}=g_2\left(\frac{x}{x_o},\frac{y}{y_o},\frac{U_\infty y_o^2}{\nu x_o}\right) \tag{23}$$

x_o can be eliminated therefore. This can be accomplished by multiplication of $\frac{U_\infty y_o^2}{\nu x_o}$ by $\frac{x_o}{x}$. The multiplication results in

$$\left(\frac{U_\infty y_o^2}{\nu x_o}\right)\left(\frac{x_o}{x}\right)=\frac{U_\infty y_o^2}{\nu x}$$

Hence 22 and 23 become

$$\frac{u}{U_\infty}=f_3\left(\frac{x}{x_o},\frac{y}{y_o},\frac{U_\infty y_o^2}{\nu x}\right) \tag{24}$$

$$\frac{v y_o}{\nu}=g_2\left(\frac{x}{x_o},\frac{y}{y_o},\frac{U_\infty y_o^2}{\nu x}\right) \tag{25}$$

The physical principle implies 24 and 25 assume significance provided the left and right hand sides of these equations are dimensionally consistent and homogeneous. There is no characteristic length in x-direction, and hence the right hand sides of 24 and 25 are independent of x_o. Thus

$$\frac{u}{U_\infty}=f_3\left(\frac{y}{y_o},\frac{U_\infty y_o^2}{\nu x}\right) \tag{26}$$

$$\frac{vy_o}{\nu} = g_3\left(\frac{y}{y_o}, \frac{U_\infty y_o^2}{\nu x}\right) \tag{27}$$

y_o can be eliminated by the multiplication of $\frac{U_\infty y_o^2}{\nu x}$ by $\frac{y_o}{y}$ in 26 and 27 and multiplying $\frac{vy_o}{\nu}$ by $\frac{y}{y_o}$ on left hand side of 27. The result is

$$\frac{u}{U_\infty} = f_4\left(\frac{y}{y_o}, \frac{U_\infty y^2}{\nu x}\right) \tag{28}$$

$$v = \frac{\nu}{v} g_4\left(\frac{y}{y_o}, \frac{U_\infty y^2}{\nu x}\right) \tag{29}$$

According to the physical principle, 28 and 29 assume significance provided that the left and right hand sides are dimensionally homogeneous.There is no characteristic length in y-direction. As a consequence, the right hand side of 28 and 29 must be independent of y_o. This process leads to eliminated expression

$$\frac{u}{U_\infty} = f_4\left(\frac{U_\infty y^2}{\nu x}\right) \tag{30}$$

$$v = \frac{\nu}{v} g_4\left(\frac{U_\infty y^2}{\nu x}\right) \tag{31}$$

30 can be rearranged as

$$\frac{u}{U_\infty} = f_5(\eta) \tag{32}$$

where η is the similarity variable and is given as

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}} \tag{33}$$

The similarity parameter contains both x and y variables, deeming it an important parameter in transforming the partial differential equations into an ordinary differential equation in one independent variable. It is relatively easy to evaluate an ordinary differential equation as compared to a partial differential equation. The similarity variable can be rearranged as

$$\frac{\nu}{y} = \frac{1}{\eta}\sqrt{\frac{\nu U_\infty}{x}} \tag{34}$$

The v-component of velocity ultimately becomes

$$v = \frac{1}{\eta}\sqrt{\frac{\nu U_\infty}{x}} g_4(\eta) \tag{35}$$

Using $g_5 = \frac{g_4}{\eta}$

$$v = \sqrt{\frac{\nu U_\infty}{x}} g_5(\eta) \tag{36}$$

Evaluating the partial differentials with respect to x and y such that

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[U_\infty f_5(\eta)] = U_\infty \frac{df_5}{d\eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial u}{\partial x} = U_\infty \frac{df_5}{d\eta}\left[-\frac{\eta}{2x}\right] \tag{37}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left[\sqrt{\frac{\nu U_\infty}{x}} g_5(\eta)\right] = \left[\sqrt{\frac{\nu U_\infty}{x}}\right]\frac{dg_5}{d\eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial v}{\partial y} = \frac{U_\infty}{x}\frac{dg_5}{d\eta} \tag{38}$$

Substituting 37 and 38 in the continuity equation

$$U_\infty \frac{df_5}{d\eta}\left[-\frac{\eta}{2x}\right] + \frac{U_\infty}{x}\frac{dg_5}{d\eta} = 0$$

Which ultimately leads to the following mathematical expression

$$\frac{dg_5}{d\eta} = \frac{\eta}{2}\frac{df_5}{d\eta} \tag{39}$$

This indicates the dependence of f_5 and g_5. The integration of above expression yields with another constant

$$g_5 = \frac{1}{2}\int \eta df_5 \tag{40}$$

Through integration by parts

$$g_5 = \frac{1}{2}\left(\eta f_5 - \int f_5 d\eta\right) \tag{41}$$

Introducing a new variable $f = \int f_5 d\eta$ and using it along with 32

$$\frac{u}{U_\infty} = \frac{df}{d\eta} \tag{42}$$

using 41 and 37 following relation can be obtained

$$\frac{v}{U_\infty} = \frac{1}{2}\sqrt{\frac{\nu}{xU_\infty}}\left(\eta\frac{df}{d\eta} - f\right) \tag{43}$$

where similarity variable η can be expressed as

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}} \tag{44}$$

42 and 43 are originally obtained by Blasius. Introduction of these expressions in momentum equation is the next step. Derivatives have to be determined initially

$$\frac{\partial u}{\partial x} = \frac{du}{d\eta}\frac{\partial \eta}{\partial x} \tag{45}$$

Using the above expression for similarity variable η and $\frac{u}{U_\infty}$, the following expressions are obtained

$$\frac{du}{d\eta} = \frac{d}{d\eta}\left(U_\infty \frac{df}{d\eta}\right)$$

$$\frac{du}{d\eta} = U_\infty \frac{d^2 f}{d\eta^2} \tag{46}$$

$$\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left(y\sqrt{\frac{U_\infty}{\nu x}}\right) = -\frac{y}{2}\left(\sqrt{\frac{U_\infty}{\nu x}}\right)\frac{1}{x}$$

$$\frac{\partial \eta}{\partial x} = -\frac{\eta}{2x} \tag{47}$$

$$\frac{\partial u}{\partial x} = -U_\infty \frac{d^2 f}{dn^2}\frac{n}{2x} = -\frac{U_\infty}{2x} n \frac{d^2 f}{dn^2}. \tag{48}$$

Next, it is possible to express $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$ in terms of η:

$$\frac{\partial u}{\partial y} = \frac{du}{dn}\frac{dn}{dy} = \frac{d}{dn}\left(U_\infty \frac{df}{dn}\right)\frac{\partial}{\partial y}\left(y\sqrt{\frac{U_\infty}{\nu x}}\right) = U_\infty\sqrt{\frac{U_\infty}{\nu x}}\frac{d^2 f}{dn^2}. \tag{49}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(U_\infty\sqrt{\frac{U_\infty}{\nu x}}\frac{d^2 f}{d\eta^2}\right) = \frac{d}{d\eta}\left(U_\infty\sqrt{\frac{U_\infty}{\nu x}}\frac{d^2 f}{d\eta^2}\right)\frac{\partial \eta}{\partial y} \\ &= \frac{d}{d\eta}\left(U_\infty\sqrt{\frac{U_\infty}{\nu x}}\frac{d^2 f}{d\eta^2}\right)\frac{\partial}{\partial y}\left(y\sqrt{\frac{U_\infty}{\nu x}}\right) = \frac{U_\infty}{\nu x}\frac{d^3 f}{d\eta^3}.\end{aligned} \tag{50}$$

Substituting 42, 43, **??**, 49 and 50 into the momentum equation following result is obtained

$$U_\infty \frac{df}{d\eta}\left(-\frac{U_\infty}{2x}\eta\frac{d^2f}{d\eta^2}\right)+\left[\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left(\eta\frac{df}{d\eta}-f\right)\right]\times\left[U_\infty\frac{d^2f}{d\eta^2}\left(\sqrt{\frac{U_\infty}{x\nu}}\right)\right] = \nu\left[U_\infty\left(\frac{U_\infty}{x\nu}\right)\frac{d^3f}{d\eta^3}\right]. \tag{51}$$

After much simplification, the Blasius equation is obtained along with boundary conditions

$$\frac{d^3f}{d\eta^3}+\frac{1}{2}f\frac{d^2f}{d\eta^2}=0 \tag{52}$$

52 is a nonlinear third order differential equation. The boundary conditions corresponding to 52 are as follows

$$\begin{aligned}\eta=0 \qquad & \frac{df}{d\eta}=0\\ \eta=0 \qquad & f=0\\ \eta=\infty \qquad & \frac{df}{d\eta}=1\end{aligned}$$

The leading edge is in fact a mathematical discontinuity in u. There is no exact analytical solution for this expression. In 1908, it was solved by Blasius using power series for small values of η and an asymptotic expansion for large values of η. Oosthuizen and Naylor (1999) presented numerical solutions. Howard (1938) gave the accurate tabulation of functions f, $\frac{df}{d\eta}$ and $\frac{d^2f}{d\eta^2}$. Maple 2022 can be used to numerically solve 52.

```
> restart;
```

#Laminar External Boundary Layer
#Solution of Velocity Boundary Layer over a Flat plate using Polhausen Similarity Solution
#η `is the dimensionless similarity variable and`$\eta=\frac{y}{\sqrt{\frac{(upsilonx)}{U}}}$

#$(\frac{u}{U})=(\frac{df}{d\eta})$

```
> Digits:=16;
```

$$Digits := 16 \tag{53}$$

```
> interface (rtablesize = 200);
```

$$[10, 10] \tag{54}$$

```
> interface(displayprecision=6);
```

$$6 \tag{55}$$

```
> beta=seq(eta,eta=0..5,0.2):
```

```
> de:=diff(f(eta),eta,eta,eta)+0.5*f(eta)*diff(f(eta),eta,eta);
```

$$de := \frac{d^3}{d\eta^3}f(\eta)+0.500000f(\eta)\left(\frac{d^2}{d\eta^2}f(\eta)\right) \tag{56}$$

```
>
```

```
> G := dsolve(
    \{de, f(0) = 0, D(f)(0) = 0, D(f)(10) = 1\},
    f(eta),
    type = numeric
):
```

```
> F:=G(2);
```

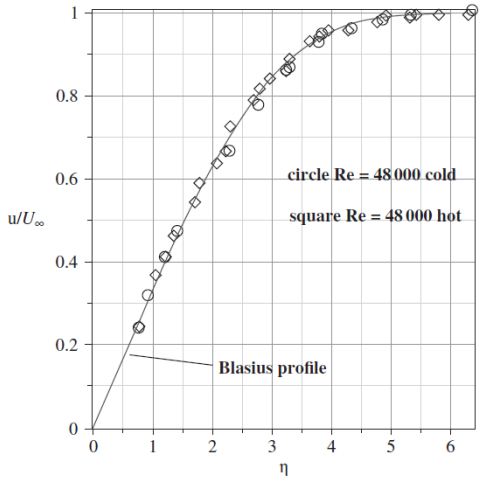
$$F := \left[\eta = 2.0, f(\eta) = 0.650025, \frac{d}{d\eta} f(\eta) = 0.629766, \frac{d^2}{d\eta^2} f(\eta) = 0.266752\right] \tag{57}$$

The above Maple code can be used for solving 52. The follwing table presents the data validation

Table 1: Blasius boundary layer solution

η	f	$\mathrm{d}f/\mathrm{d}\eta$	$\mathrm{d}^2f/\mathrm{d}\eta^2$
0.000000	0.000000	0.000000	0.332057
0.200000	0.006641	0.066408	0.331984
0.400000	0.026560	0.132764	0.331470
0.600000	0.059735	0.198937	0.330079
0.800000	0.106108	0.264709	0.327389
1.000000	0.165572	0.329780	0.323007
1.200000	0.237949	0.393776	0.316589
1.400000	0.322982	0.456262	0.307865
1.600000	0.420321	0.516757	0.296663
1.800000	0.529518	0.574758	0.282931
2.000000	0.650025	0.629766	0.266752
2.200000	0.781193	0.681310	0.248351
2.400000	0.922290	0.728982	0.228092
2.600000	1.072506	0.772455	0.206455
2.800000	1.230977	0.811510	0.184007
3.000000	1.396808	0.846045	0.161360
3.200000	1.569095	0.876082	0.139128
3.400000	1.746950	0.901761	0.117876
3.600000	1.929525	0.923330	0.098086
3.800000	2.116030	0.941118	0.080126
4.000000	2.305747	0.955518	0.064234
4.200000	2.498040	0.966957	0.050520
4.400000	2.692361	0.975871	0.038973
4.600000	2.888248	0.982683	0.029484
4.800000	3.085321	0.987789	0.021871
5.000000	3.283274	0.991542	0.015907

Interpretation of above in Maple 2022 results in the following plot



H

Figure 5: Comparison of longitudinal velocity profile with experimental data of Junkhan and Seroyvy (1966, 1967) (Sourced from Onur (2023))

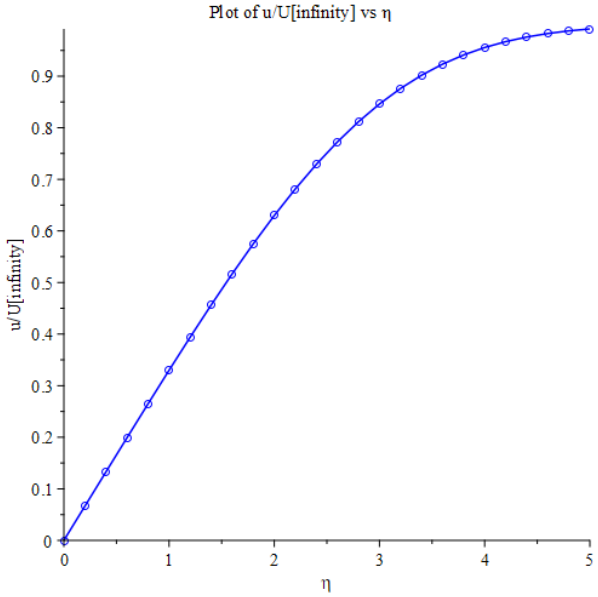


Figure 4: Numerical Solution of Blasius Equation 52

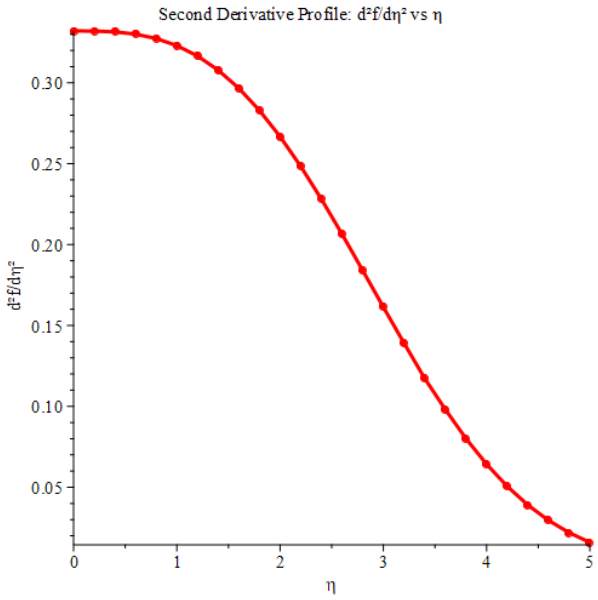


Figure 6: Second Derivative Profile: d^2f/d^2 vs. η

x-Component of velocity $\frac{u}{U_\infty}$

In reference with 1, the variation in longitudinal velocity $\frac{u}{U_\infty} = \frac{df}{d\eta}$ as function of η is given in 4. The comparison of numerical data provided by Junkhan and Seroyvy (1966) and Junkhan and Seroyvy (1967) is coincident with the numerical data simulated above. The close agreement between theory and experiment is evident.

Boundary Layer Thickness $\delta(x)$

Boundary layer thickness $\delta(x)$ is defined as the distance y from the plate where the velocity ratio $u/U_\infty = 0.99$. In other words, the boundary layer thickness $\delta(x)$ is the location where the velocity u is 99% of free stream velocity U_∞. From Table 7.1, we see that $u/U_\infty = 0.99$ when $\eta \equiv 5.0$, and since $\eta \equiv y\sqrt{U_\infty/\nu x}$, we have

$$5.0 \approx \delta(x)\sqrt{\frac{U_\infty}{\nu x}}. \tag{58}$$

We can now express the boundary layer thickness $\delta(x)$ in the following form:

$$\frac{\delta(x)}{x} = \frac{5}{\sqrt{\mathrm{Re}_x}} \tag{59}$$

where Re_x is the local Reynolds number and is defined as $\mathrm{Re}_x = U_\infty x/\nu$. Boundary layer thickness $\delta(x)$ is useful in gauging the influence of viscous diffusion.

Wall Shear Stress τ_w

The Blasius solution may also be used to calculate the drag on a flat plate. We can evaluate the wall shear stress as follows:

$$\tau_w(x) = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}}\frac{d^2f}{d\eta^2}\bigg|_{\eta=0} \tag{60}$$

where the gradient of u is obtained from 49 . From 1, one may get $f''(0) = 0.332$, and consequently, 60 becomes

$$\tau_w(x) = 0.33206\mu U_\infty \sqrt{\frac{U_\infty}{\nu x}} \tag{61}$$

or

$$\tau_w(x) = \frac{0.33206\mu U_\infty}{x}\sqrt{\mathrm{Re}_x} \tag{62}$$

where $\mathrm{Re}_x = U_\infty x/\nu$ is the local Reynolds number.

Local Skin Friction Coefficient $c_f(x)$

We can obtain the dimensionless local skin friction coefficient as given below

$$c_f = \frac{\tau_w}{\rho U_\infty^2/2} = \frac{2\tau_w}{\rho U_\infty^2} = \frac{2\left[\frac{0.33206\mu U_\infty}{x}\sqrt{\mathrm{Re}_x}\right]}{\rho U_\infty^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} \tag{63}$$

Drag Force D

The drag force D on one side of the plate assuming that the solution is valid from the leading edge is obtained as follows. Assuming a depth of b units into paper, the drag force is obtained using 60

$$D = \int_{x=0}^{L} \tau_w(x)(b\,dx) = \mu b\, U_\infty \sqrt{\frac{U_\infty}{\nu}}\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} \int_{x=0}^{L} \frac{dx}{\sqrt{x}} = \mu U_\infty \sqrt{\frac{U_\infty}{\nu}}\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} 2\sqrt{L} \tag{64}$$

or

$$D = 0.664\mu b\, U_\infty \sqrt{\mathrm{Re}_L} \tag{65}$$

where $\mathrm{Re}_L = \rho U_\infty L/\mu$ is the mean Reynolds number and b is the width of the plate.

Average Skin Friction Coefficient $\overline{c_f}$

It is customary to express drag data in terms of a mean drag coefficient $\overline{c_f}$. The mean skin friction coefficient $\overline{c_f}$ is defined as

$$\overline{c_f} = \frac{D}{\rho U_\infty^2 A/2} = \frac{0.664\mu b\, U_\infty \sqrt{\mathrm{Re}_L}}{\rho U_\infty^2 bL/2} = \frac{1.328}{\sqrt{\mathrm{Re}_L}} \tag{66}$$

where $\mathrm{Re}_L = U_\infty L/\nu$ is the mean Reynolds number based on the entire plate length L. In other words, one has to first find the total shear force experienced by the surface of axial length L

$$L\overline{\tau_w} = \int_0^L \tau_w(x)\,dx \tag{67}$$

where $\overline{\tau_w}$ is the average shear stress. The average skin friction coefficient $\overline{c_f}$ is

$$\overline{c_f} = \frac{\overline{\tau_w}}{\rho U_\infty^2/2} \tag{68}$$

Displacement Thickness $\delta_1(x)$

$$\frac{\delta_1(x)}{x} = \frac{1}{\sqrt{\mathrm{Re}_x}} \int_0^\infty \left[1 - \frac{df}{d\eta}\right] d\eta \tag{69}$$

and for the flow being analyzed using integration command trapz in MATLAB 2021a:

```
trapz(eta, 1-fp)
ans =
1.7221
```

Therefore, the displacement thickness $\delta_1(x)$ is

$$\frac{\delta_1(x)}{x} = \frac{1.7221}{\sqrt{\mathrm{Re}_x}} \tag{70}$$

Momentum Thickness $\delta_2(x)$

$$\frac{\delta_2(x)}{x} = \frac{1}{\sqrt{\mathrm{Re}_x}} \int_0^\infty \left(\frac{df}{d\eta}\right)\left[1 - \frac{df}{d\eta}\right] d\eta \tag{71}$$

and for the flow being analyzed using integration command trapz in MATLAB 2021a:

```
>> F = fp.*(1-fp);
>> trapz(eta, F)
ans =
0.6620
```

Momentum thickness $\delta_2(x)$ can be given as

$$\frac{\delta_2(x)}{x} = \frac{0.6620}{\sqrt{\mathrm{Re}_x}} \tag{72}$$

Conclusion

The application of similarity transformation has successfully simplified the analysis of laminar boundary layer flow over a semi-infinite flat plate, reducing the governing PDEs to a manageable ODE. The numerical solution of the resulting Blasius equation accurately predicts the velocity distribution within the boundary layer and agrees well with experimental data. Key boundary layer properties, including boundary layer thickness, wall shear stress, and local skin friction were computed and expressed as functions of the local Reynolds number. The analysis confirms that at ≈ 5.0, the velocity reaches 99% of the freestream, aligning with theoretical expectations. The dimensionless results, such as the wall shear coefficient $f^{"}(0) = 0.332$ also correlates to classical literature. This study not only validates the theoretical framework proposed by Blasius but also demonstrates the effectiveness of modern symbolic computation tools like Maple for solving complex fluid flow problems.

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